Sec 19.1 to 19.4 — Course

Chap 19. Linear programming

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Linear programming is the name used for problems in which the objective is to maximise or minimise a linear function subject to linear inequality constraints.

GENERAL LINEAR PROGRAMMING PROBLEM
(WITH ONLY TWO DECISION VARS.)

Max/Min

$$z = c_1 x_1 + c_2 x_2$$ (CRITERION FUNCTION) OR OBJECTIVE FUNCTION.

subject to m constraints

$$a_{11} x_1 + a_{12} x_2 \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 \leq b_2$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 \leq b_m$$

(INEQUALITY CONSTRAINTS)

and $x_1 \geq 0$, $x_2 \geq 0$. (NON-NEGATIVITY CONSTRAINTS)

HAS A GRAPHICAL SOLUTION

(x_1, x_2) pairs for which all the constraints are satisfied are called Feasible Region ~~constraints~~. The point of intersection of any two or more border is called EXTREME POINTS.

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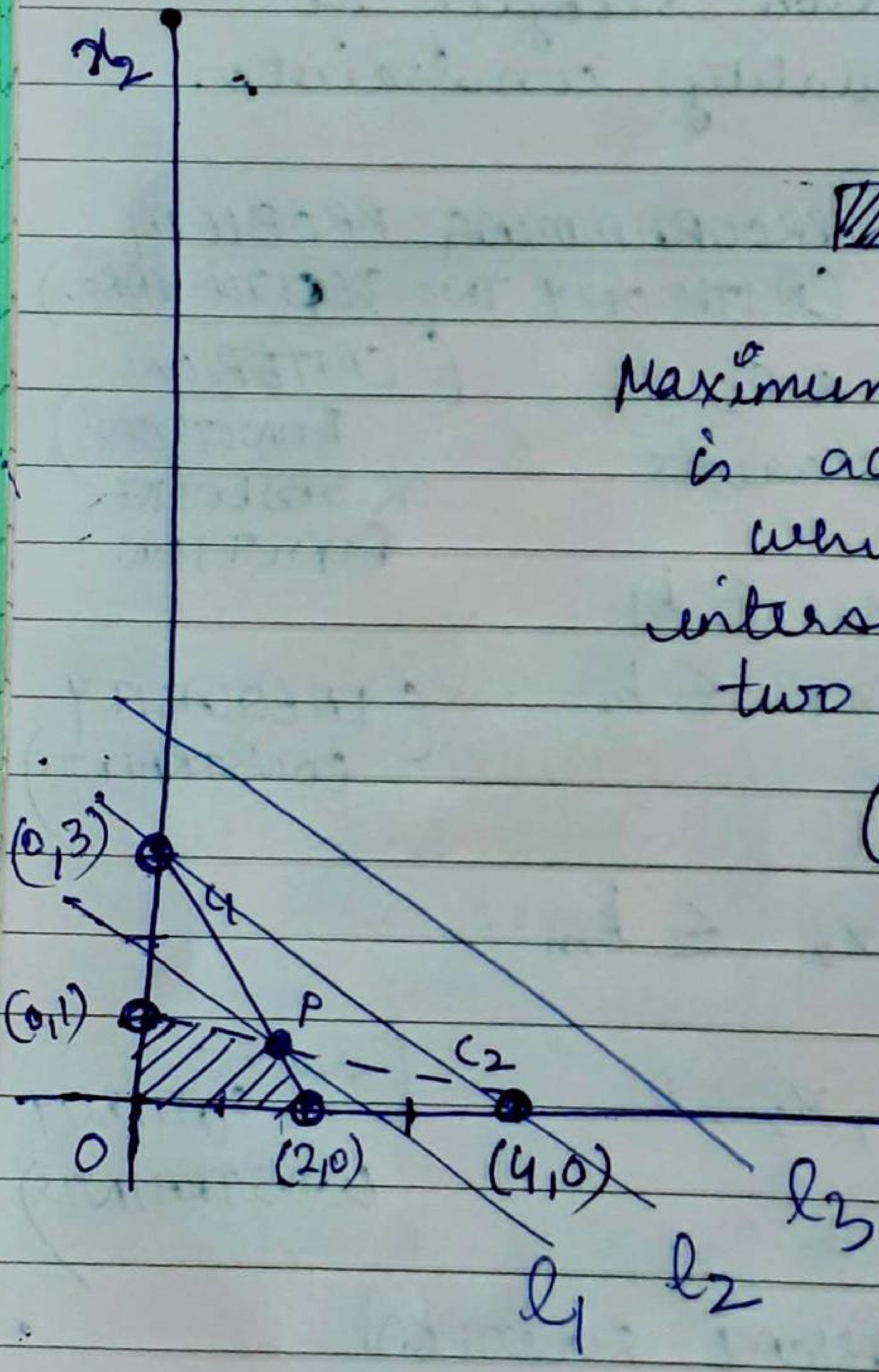
Q1 HAS

a) Max $z = 3x_1 + 4x_2$ s.t

$$\begin{cases} 3x_1 + 2x_2 \le 6 \rightarrow C_1 \\ x_1 + 4x_2 \le 4 \rightarrow C_2 \end{cases} \quad x_1, x_2 \ge 0$$



Maximum objective f^n is achieved at P which is pt of intersection of two constraints (1.6, 0.6)

l_1, l_2, l_3 are the level curves of $z = 3x_1 + 4x_2$

Explanation of Solution

In the above problem, the objective function is:

$Z = 3x_1 + 4x_2$. Here x_1 and x_2 are the decision variables

We, then plot the two inequalities given. Since the constraints are "less than or equal to" constraints, the shaded region ▨ below the two lines is the feasible region. The solution to this linear programming problem must lie within the shaded region.

Recall that the solution is a point (x_1, x_2) such that the value of Z is the largest it can be, while still lying in the feasible region.

Since $Z = 3x_1 + 4x_2$, plotting the line $x_2 = \frac{Z}{4} - \frac{3}{4}x_1$

results in level curves, which have the same slope. The level curves are parallel to each other and the

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line l_1 intersects the feasible region at the highest possible level of z which is
$3(1.6) + 4(0.6) = 7.2$. ∴,

$Z = 7.2$ is the highest possible value of z given the constraint. This value occurs at the intersection of the lines $3x_1 + 2x_2 = 6$ and $x_1 + 4x_2 = 4$ where $x_1 = 1.6$ and $x_2 = 0.6$.

Q4 H&S A firm produces two types of television set an inexpensive type (A) and an expensive type (B). The firm earns a profit of 700 from each type A TV, and 1000 for each TV of type B. There are three stages of the production process. Stage I requires 3 hours of labor on each set of type A and 5 hours of labour on each set of type B. The

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total available number of hours is 3900. Stage II requires 1 hour of labour on each set of type A and 3 hours on each set of type B. The total labor they have is 2100 hours. At stage III, 2 hours of labor are needed for both types and 2200 hours of labor are available. How many TV sets of each type should the firm produce to maximize its profit?

Solⁿ The problem can be formulated as that of linear programing. We have to

Max $700x_1 + 1000x_2$ subject to the following constraints:

$C_1: \quad 3x_1 + 5x_2 \leq 3900$

$C_2: \quad x_1 + 3x_2 \leq 2100$

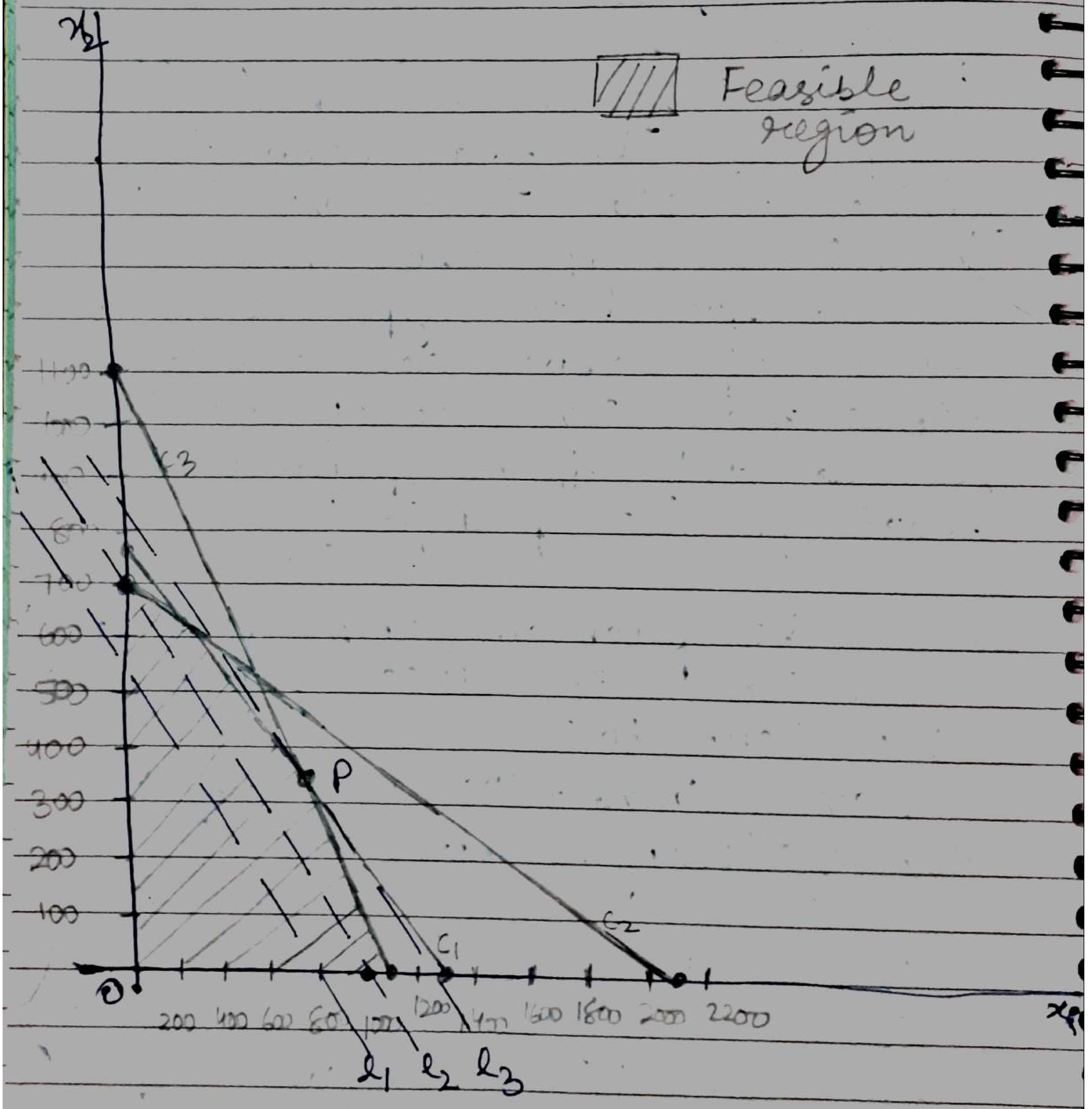
$C_3: \quad 2x_1 + 2x_2 \leq 2200$

along with non-negativity constraints

$x_1 \geq 0, \quad x_2 \geq 0.$

Where x_1 is no. of TV sets of type A and x_2 is no. of TV sets of type B

Graphical Solⁿ



Objective f^n level curves :

$$x_2 = \frac{Z}{1000} - 0.7x_1$$

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we have plotted the 3 constraints and some level curves of the objective function. The objective function is maximised at the point P in the feasible region. ∴ Max z is achieved at (800, 300) which is the point of intersection of the constraints C_1 and C_3. ∴, to maximise profits 800 TV sets of type A and 300 TV sets of type B should be produced.

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(H&S example 19.1)

Ques A baker has 150 kg of flour, 22 kg of sugar and 27.5 kg of butter with which to make two types of cake. Suppose that making of one dozen A cakes requires 3 kilos of flour, 1 kilo of sugar, and 1 kilo of butter whereas making one dozen B requires 6 kilo of flour, 0.5 kilo of sugar and 1 kilo of butter. Suppose that profit from one dozen A cakes is 20 and from one dozen B cakes is 30. How many dozen A cakes (x_1) and how many dozen B cakes (x_2) will maximise the baker's profit?

Sol^n Obj f^n: Max $20x_1 + 30x_2$

S.t.

C_1: $3x_1 + 6x_2 \le 150$ flour const^t

C_2: $x_1 + 0.5x_2 \le 22$ Sugar const^t

C_3: $x_1 + x_2 \le 27.5$ butter const^t

along with non-negativity constraint:

$x_1 \ge 0$ and $x_2 \ge 0$

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Graphical Solution

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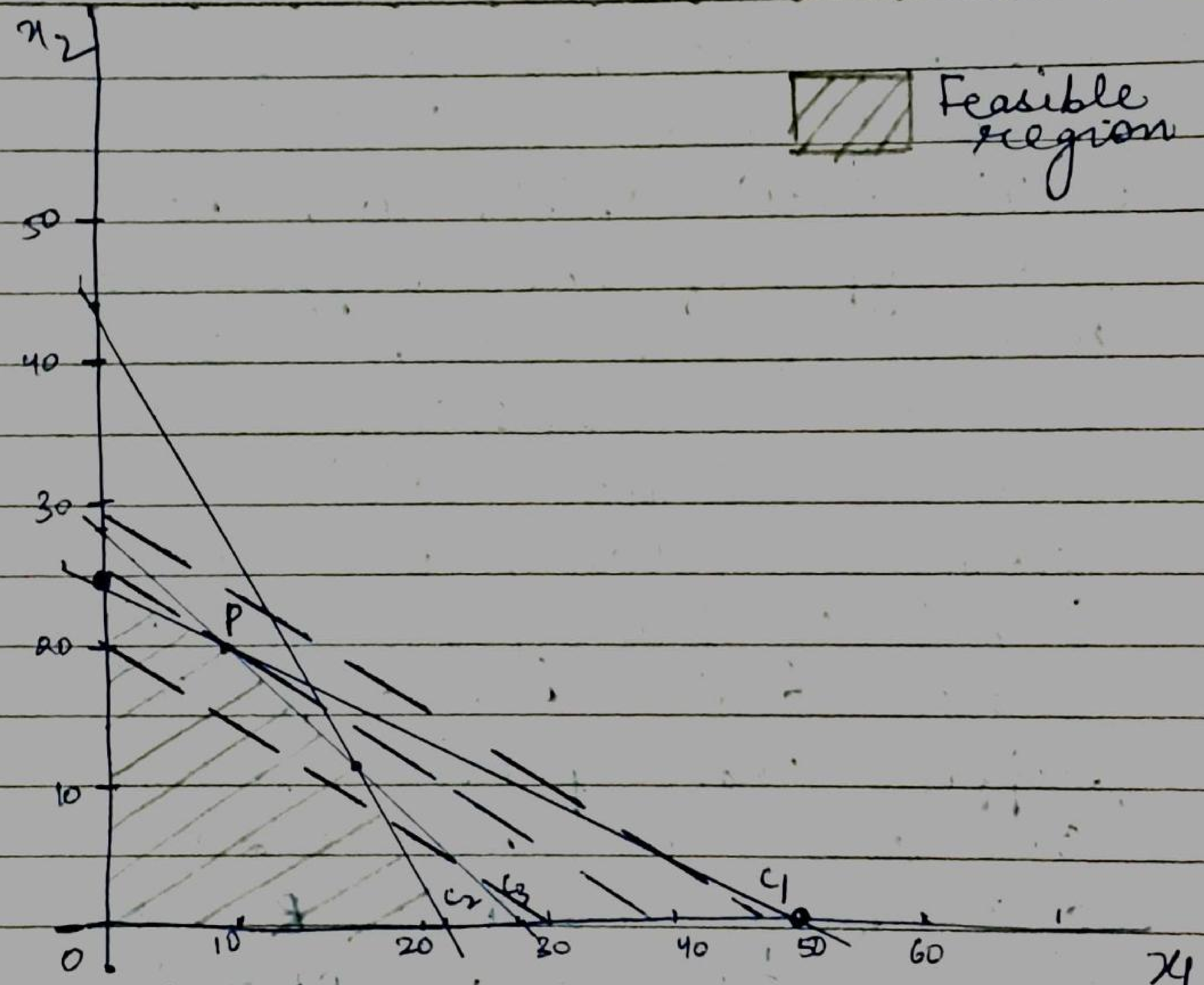

$$Z = 20x_1 + 30x_2$$

Level curves: $x_2 = \frac{Z}{30} - \frac{20}{30}x_1$

P is the optimal point where c_1 and c_3 are binding. ∴ P is the point of intersection of c_1 and c_3 ⇒ P is (5, 22.5)

Profit $= Z^* = 20(5) + 30(22.5)$
$= 775$

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DUALITY THEORY

Duality theory of linear programming answers the question that what happens to the optimal solution if the availability of the resource changes.

For example, consider the baker's problem and let the baker recieve free of cost, one extra kilo of flour or an extra kilo of sugar or an extra kilo of butter. In each of these situation let us calculate the marginal profits to the baker.

Case I Extra kilo of flour

c_1'. New constraint: $3x_1 + 6x_2 \le 151$

New optimal: Intersection of c_1' and c_3 (since c_2 was not binding $\therefore$ at the old optimal sugar was unused)

$\Rightarrow$ Solving c_1' and c_3 we get

$$(x_1, x_2) = \left(\frac{14}{3}, \frac{137}{6}\right)$$

$$\text{New } z^{*'} = 20(14/3) + 30(137/6)$$

$$= 775 + 10/3$$

$$\Delta \text{Profit} = 10/3 = u_1^*$$

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Case II Extra kilo of sugar

New constraint $C_2' \Rightarrow x_1 + 0.5x_2 \leq 23$

optimal point P will not change though feasible set will expand.

$$\Delta \text{profit} = 0 = u_2^*$$

Case III Extra kilo of butter

New constraint $C_3' \Rightarrow x_1 + x_3 \leq 28.5$

New optimal point is the intersection of C_1 and C_3'.

$\Rightarrow$ Solving C_1 and C_3' we get

$$(x_1, x_2) = (7, 21.5)$$

$$\text{New } z^* = 20(7) + 30(21.5)$$
$$= 775 + 10.$$

$$\Delta \text{profit} = 10 = u_3^*$$

Dual Problem: Suppose the baker doesnot want to continue with the business and wants to sell all the ingredients to someone else. He intends to charge a price u_1 for each kilo of flour, u_2 for each kilo of sugar and u_3 for each kilo of butter. Since he is foregoing the profit from producing and selling cakes, he wants that u_1, u_2 and u_3 should be such that :

$$\underbrace{3u_1 + u_2 + u_3}_{\text{when price charged } (u_1, u_2, u_3) \text{ this is the earnings from ingredients needed to produce 1 cake of type A}} \geq \underbrace{20}_{\text{profit from selling 1 cake of type A}}$$

Similarly, $6u_1 + 0.5u_2 + u_3 \geq 30$

Now, total cost to the buyer of the business is :

$150u_1 + 22u_2 + 27.5u_3$

The intended buyer's problem is :

Min $150u_1 + 22u_2 + 27.5u_3$

subject to

$$3u_1 + u_2 + u_3 \geq 20$$

$$6u_1 + 0.5u_2 + u_3 \geq 30$$

with the non-negativity constraints

$u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq 0$.

→ called the DUAL PROBLEM.

ORIGINAL PROBLEM – called PRIMAL PROBLEM

Solution to the buyer's problem

$u_1^* = 10/3, \quad u_2^* = 0, \quad u_3^* = 10$

cost to the buyer $= 150(10/3) + 22(0)$
$+ 27.5(10)$
$= 775$

This is maximum value of the criterion function of the original problem.

RESULTS OF DUALITY THEORY

General LP problem can be written as:

max $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$
subject to

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n \le b_1 \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n \le b_m \end{cases}$$

with non-negativity constraints
$x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0.$

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The dual of the LP problem is:

min $b_1 u_1 + \dots + b_m u_m$ subject to

$$\begin{cases} a_{11} u_1 + \dots + a_{m1} u_m \geq c_1 \\ \vdots \\ a_{1n} u_1 + \dots + a_{mn} u_m \geq c_n \end{cases}$$

with non-negativity constraints
$u_1 \geq 0, \dots, u_m \geq 0$. [2]

[1] is referred to as the Primal problem with n variables $x_1, x_2, \dots, x_n$ and m constraints. [2] is called the Dual problem with m variables $u_1, u_2, \dots, u_m$ and n constraints.

Let us introduce the following column vectors:

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A' = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

<u>Primal</u> Max $c'x$ s.t. $Ax \le b$
with $x \ge 0$

<u>Dual</u> Min $b'u$ s.t. $A'u \ge c$
with $u \ge 0$

$\boxed{\text{OR}}$ Min $u'b$ s.t. $u'A \ge c'$
with $u \ge 0$

[Because $b'u = u'b$ and $A'u \ge c$ is equivalent to $u'A \ge c'$]

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DUALITY THEORY

RESULT I

If $(x_1, x_2, \ldots, x_n)$ is feasible in the primal problem and $(u_1, u_2, \ldots, u_m)$ is feasible in the dual problem, then

$$b_1 u_1 + b_2 u_2 + \ldots + b_m u_m \geq c_1 x_1 + \ldots + c_n x_n$$

i.e. the dual criterion function has a value that is always at least as large as that of the primal.

Result II

Suppose that $(x_1^*, x_2^*, \ldots, x_n^*)$ and $(u_1^*, u_2^*, \ldots, u_m^*)$ are feasible in problems [1] and [2] respectively and that

$$c_1 x_1^* + \ldots + c_n x_n^* = b_1 u_1^* + \ldots + b_m u_m^*$$

Then $(x_1^*, \ldots, x_n^*)$ solves problem [1] and $(u_1^*, u_2^*, \ldots, u_m^*)$ solves problem [2].

So, if we are able to find feasible solutions for problems [1] and [2] that give the same value to the criterion function in each of the two problems, then these feasible solutions are optimal solutions.

Result III (Duality Theorem)

Suppose the primal problem has a (finite) optimal solution. Then the dual problem also has a (finite) optimal solution and the corresponding value of the criterion functions are equal. If the primal has no bounded optimum, then the dual has no feasible solution.

How to write dual of given LP problem

If we are given LP problem as follows :

$$\text{Max} \quad Z = 40x_1 + 60x_2$$
$$\text{S.t} \quad 6x_1 + 9x_2 \le 180$$
$$2x_1 + 1.5x_2 \le 48$$
$$x_1, x_2 \ge 0$$

In order to construct the dual problem of this given problem, (which we call the PRIMAL) we need to keep the following points in mind.

① If the objective of the primal is Maximisation then objective of the dual shall be reversed (i.e it will be minimisation)

② Sign of the constraints in the primal are " $\le$ ", in the

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dual problem, these signs will be reversed and become "$\geq$".

(3) The number of constraints in the primal will be equal to the number of decision variables in the dual problem.

(4) The number of decision variables in the primal will be equal to the number of constraints in the dual problem.

(5) The coefficients of the variables in the objective function of the primal are equal to RHS constants of the constraints of the dual problem and vice versa.

Using these points, if we let the decision variables of the dual problem to be equal to u_1 and u_2, we get the following dual problem:-

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Min $z = 180 u_1 + 48 u_2$

subject to

$6 u_1 + 2 u_2 \geq 40$

$9 u_1 + 1.5 u_2 \geq 60 ; u_1, u_2 \geq 0$

Please not that coefficients of u_1 and u_2 (namely 6 and 2) in constraint 1 are the coefficients of x_1 ~~and~~ in constraint 1 and constraint 2 respectively.

The coefficient matrix of the LHS of constraints of the dual

$$\begin{bmatrix} 6 & 2 \\ 9 & 1.5 \end{bmatrix}$$

is the transpose of coefficient matrix of the LHS of constraints of the primal

$$\begin{bmatrix} 6 & 9 \\ 2 & 1.5 \end{bmatrix}$$

Now, suppose we are given minimisation problem

Min $Z = 20 x_1 + 12 x_2$

s.t. $10 x_1 + 25 x_2 \geq 2400$

$40 x_1 + 25 x_2 \geq 3600$

$x_1 \geq 0, \ x_2 \geq 0$

Following similar guidelines, we get the dual as :

$$\text{Max } z = 2400 u_1 + 3600 u_2$$

s.t

$$10 u_1 + 40 u_2 \leq 20$$
$$25 u_1 + 25 u_2 \leq 12$$
$$u_1, u_2 \geq 0$$

Ex. <u>Note</u>: Remember that Dual of Dual is the Primal.

EXPLANATION OF DUALITY THEOREMS

The Result I compares the feasible solutions of the primal and dual problem. It says that objective function value of the feasible solutions of the dual problem is always greater than or equal to objective function values of the feasible solution of the primal. Consider a given LP problem

$$\text{Max } 3x_1 + 4x_2$$

s.t. $3x_1 + 2x_2 \leq 6$

$x_1 + 4x_2 \leq 4$

$x_1, x_2 \geq 0$

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Let us consider some feasible solutions of the primal and the corresponding values of objective function :

$Z(0,0) = 0$
$Z(0,1) = 4$
$Z(1,0) = 3$
$Z(1, 0.5) = 5$
$Z(2,0) = 6$
$Z(1.6, 0.6) = 7.2$

Now, let us write the dual of the original problem

$$\text{Min } 6u_1 + 4u_2$$
$$\text{s.t. } 3u_1 + u_2 \geq 3$$
$$2u_1 + 4u_2 \geq 4$$
$$u_1, u_2 \geq 0$$

Again, we shall write few feasible solution and corresponding objective function value :

$Z(2,3) = 24$
$Z(3,1) = 22$

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$z(2,2) = 20$
$z(2,1) = 16$
$z(1,1) = 10$
$z(0.8, 0.6) = 7.2$

One can see that all feasible solutions of the dual have an objective f^n value which is greater than or equal to all objective f^n values of feasible solutions of the primal.
Though infinite feasible solutions exist for both the primal and dual we have just checked Result I for a set of values of objective function of primal and dual.
The way we have constructed the dual has resulted in this. We start by finding an upper estimate of the objective function for every feasible solution. So, automatically every feasible solution of the dual for a maximisation primal will have an objective function value greater than or equal to the optimum solution of the primal. Primal

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being a maximisation problem, the optimum solution of objective function value will be $\geq$ every feasible solution of the objective function value. So, we will realize that every feasible solution to the dual will have an objective function value $\geq$ that of every feasible solution of the primal.

Result II relates the optimum solutions of the primal and the dual problem.

Consider the same example we have taken to explain Result I, we see the following. (1.6, 0.6) is feasible for primal and (0.8, 0.6) is feasible for dual and both of these give the same value of the objective function equal to 7.2. According to Result II, (1.6, 0.6) is the solution of the primal problem and (0.8, 0.6) is the solution of the dual.

We can see why this Result hold by assuming that on the contrary that (1·6, 0·6) is not the optimum for primal and there is another feasible solⁿ say $(x_1^{\#}, x_2^{\#})$ which gives the objective fⁿ value say 8 ie. $z(x_1^{\#}, x_2^{\#}) = 8$. But this contradicts Result I, which says that all feasible solutions of the dual gives objective function value greater than or equal to that of primal. So no such $(x_1^{\#}, x_2^{\#})$ can exist and (1·6, 0·6) ∴ is the optimum.

Result III is somewhat similar to Result II. It says that if the primal has a finite optimal solution, then the dual also has a finite optimal solution and the corresponding values of the objective functions shall be equal.

In our example, the primal had a solution (1·6, 0·6), and the dual had a solution

(0.8, 0.6) as the optimum solution Result III holds because the objective function value of the two are equal to 7.2.

ECONOMIC INTERPRETATIONS OF THE LP PROBLEM AND ITS DUAL

Consider the general LP problem which we wrote before also:-

Max $c_1x_1 + c_2x_2 + \dots + c_nx_n$
subject to

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$

Let us put a context to the above LP problem.

$(x_1, x_2, \dots, x_n)$ are n different activities which a firm is involved in using m different resources as inputs.

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a_{ij} is the number of units of resource i that is needed to run activity 'j' at unit level.

$\therefore$ vector $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ represents the

m different total resource requirement to run activity 'j' at unit level.

If activities are run at levels $(x_1, x_2, \ldots, x_n)$, then

Total Resource req^n of m inputs $= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}$

OR

$$\left.\begin{aligned} \text{Input } 1 &= x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ \text{Input } 2 &= x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} \\ &\vdots \\ \text{Input } m &= x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn} \end{aligned}\right\}$$

$\rightarrow$ Demand of m resources

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Now availability of m resources / SS of m resources = $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

We must have:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

which are m constraints in LP problem showing that only those $x_1, x_2, \dots, x_n$ which satisfy those constraints are feasible. Also, we must have $x_1, x_2, \dots, x_n \geq 0$.

Let C_j measure the reward ~~stating~~ which producer gets by running the activity at unit level. So Total reward obtained by running activities at $(x_1, x_2, \dots, x_n)$ level is:

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

which is the objective function of the LP problem.

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So, the LP problem is:

Find those levels for the n activities that maximises total reward, subject to the given resource constraints.

DUAL INTERPRETATION

Let u_j be the relative contribution that one unit of resource j makes to total economic result. u_j doesnot represent the real market prices, so, they are called SHADOW PRICES.

The Dual problem was:

Min $b_1 u_1 + b_2 u_2 + \dots + b_m u_m$

subject to

$a_{11} u_1 + \dots + a_{m1} u_m \geq c_1$

$\vdots$

$a_{1n} u_1 + \dots + a_{mn} u_m \geq c_n$

along with

$u_1, u_2, \dots, u_m \geq 0$

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The producer has to pay for the resources used in running the n activities.

Since $(a_{11}, a_{21}, \ldots, a_m)$ represent the total input requirement of n resources needed to produce one unit of activity x_1, while the reward from one unit of x_1 is c_1, the net profit to the seller of these resources must be ≤ 0

Net profit of running activity j at unit level

$c_j - (a_{1j} u_1 + a_{2j} u_2 + \ldots + a_{mj} u_m)$ is called shadow profit.

$b_1 u_1 + b_2 u_2 + \ldots + b_m u_m$

is total shadow value of the initial stock of resources.

$\therefore$, the Dual problem is:

Find those shadow prices $u_1, u_2, \ldots, u_m$ ~~such~~ which

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minimise the shadow value of the initial resources such that profit of running each activity at unit level is ≤ 0.

we next understand that optimal dual variables are the shadow prices:

Consider primal: max $c_1x_1 + \ldots + c_nx_n$

subject to

$$a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1$$

$$\vdots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

with $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Let $b_1, b_2, \ldots, b_m$ change by the quantities $\Delta b_1, \Delta b_2, \ldots, \Delta b_m$.

Let the optimal vector be $(x_1^*, x_2^*, \ldots x_m^*)$ when RHS vector is $(b_1, b_2, \ldots, b_m)$ and $(x_1^* + \Delta x_1, \ldots, x_n^* + \Delta b_m)$ when RHS vector is $(b_1 + \Delta b_1, \ldots, b_m + \Delta b_m)$.

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If $\Delta b_1, \Delta b_2, \ldots, \Delta b_m$ are all sufficiently small, then the optimal solution of the duals in the old and new situation, shall be same namely $u_1^*, u_2^*, \ldots, u_m^*$. According to Result II, we will have

$$c_1 x_1^* + c_2 x_2^* + \cdots + c_n x_n^* = b_1 u_1^* + \cdots + b_m u_m^* \quad \text{(1)}$$

and

$$c_1(x_1^* + \Delta x_1) + \cdots + c_n(x_n^* + \Delta x_n) = (b_1 + \Delta b_1) u_1^* + \cdots + (b_m + \Delta b_m) u_m^* \quad \text{(2)}$$

Subtract (1) from (2) we get

$$\underbrace{c_1 \Delta x_1 + \cdots + c_n \Delta x_n}_{} = u_1^* \Delta b_1 + \cdots + u_m^* \Delta b_m$$

change in criterion f^n when $b_1, \ldots, b_m$ changes by $\Delta b_1, \ldots, \Delta b_m$

OR

$$\boxed{\Delta z^* = u_1^* \Delta b_1 + \cdots + u_m^* \Delta b_m}$$

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