(Sec 21.1, 21.5, 21.6, 21.7)

Differential Equations

No.

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INTRODUCTION (Sec 21.1)

An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.

Example:

1. $\dot{x} = x + t$
2. $\dot{x} = t^2 - 1$
3. $\dot{x} = t + \frac{tx}{t^2 - 1}$
4. $3\ddot{x} - 30\dot{x} + 75x = 2t + 1$

where $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$

Order of a differential equation is the order of the highest order derivative appearing in the equation.

for example, differential equation $\frac{dy}{dt} + 2y = 6$ is of order 1

However, $y''(t) + y'(t) - 2y = -10$ is of order 2.

Degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.
For instance, the differential equation $\frac{d^2Y}{dt^2} = kY$ is a second order differential equation and is of degree 1.

We shall confine to linear differential equation.

A linear differential equation of order n is of the form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}\frac{dy}{dt} + a_n y = b.$$

This differential equation is linear since all the derivatives as well as the dependent variable appear only in the first degree and no product occurs in which y and any of its derivatives are multiplied together.

A differential equation will be non- linear if :

① it's degree is more than 1
② any of the differential coefficient has exponent more than 1
③ exponent of dependent variable is more than 1
④ product containing dependent variable and its differential coefficient is present.

Solution of a differential equation

solution of a differential equation is a relation b/w variables involved which satisfies the differential equation. Such a relation and derivatives obtained

So, when substituted in the differential equation, make LHS and RHS identically equal.
The solution containing as many arbitrary constants as the order of the differential equation is called the GENERAL SOLUTION of the differential equation.
Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a PARTICULAR SOLUTION (graph of which is called INTEGRAL CURVE)

FIRST ORDER DIFFERENTIAL EQUATION.

$$\dot{x}(t) = F(t, x)$$

SOLUTION OF A DIFFERENTIAL EQ.^{n}

A differentiable function $x = x(t)$ that is defined in an open interval I of the real line and satisfies $\dot{x} = F(t, x)$ $\forall\, t \in I$ is called a Solution of $\dot{x} = F(t, x)$ in I and graph of any solution is called Integral curve

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SOLVING DIFFERENTIAL EQUATIONS

(1) $\dot{x} = \frac{t}{t^2+1}$

$\frac{dx}{dt} = \frac{t}{t^2+1} \Rightarrow \int dx = \int \frac{t}{t^2+1} dt$

$\Rightarrow x(t) = \frac{1}{2} \int \frac{2t}{t^2+1} dt$

$x(t) = \frac{1}{2} \ln |t^2+1| + C$

defined for $\forall t \in \mathbb{R}$

(2) $(e^t + e^{-t}) \frac{dy}{dt} = e^t - e^{-t}$

$dy = \frac{e^t - e^{-t}}{e^t + e^{-t}} dt$

$\int dy = \int \left(\frac{e^t - e^{-t}}{e^t + e^{-t}} \right) dt$

$y = \ln |e^t + e^{-t}| + C$

defined for $\forall t \in \mathbb{R}$

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INITIAL VALUE PROBLEM

When a differential equation is given to us and the value of the dependent variable is given at time t = 0 (called the initial condition), it constitutes an initial value problem

example Solve

$$\dot{x}(t) = x(t) + t \text{ and } x(0) = 1$$

Such initial value problems arise in many economic models. for example, suppose an economic growth model (Solow model, say) gives a differential equation for the accumulation of capital overtime. If the initial stock of capital is given, it constitutes an initial value problem, whose solution can be found.

HRS

Q2. Consider the differential equation $t\dot{x} = 2x$

a) Show that $x = ct^2$ is a solⁿ for all choices of constant c.

b) Find in particular the integral curve through (1,2)

Solⁿ a) $x = ct^2$

$\dot{x} = 2ct$

$t\dot{x} = 2ct^2 = 2x$

Hence, proved

b) Integral curve is graph of the solution of

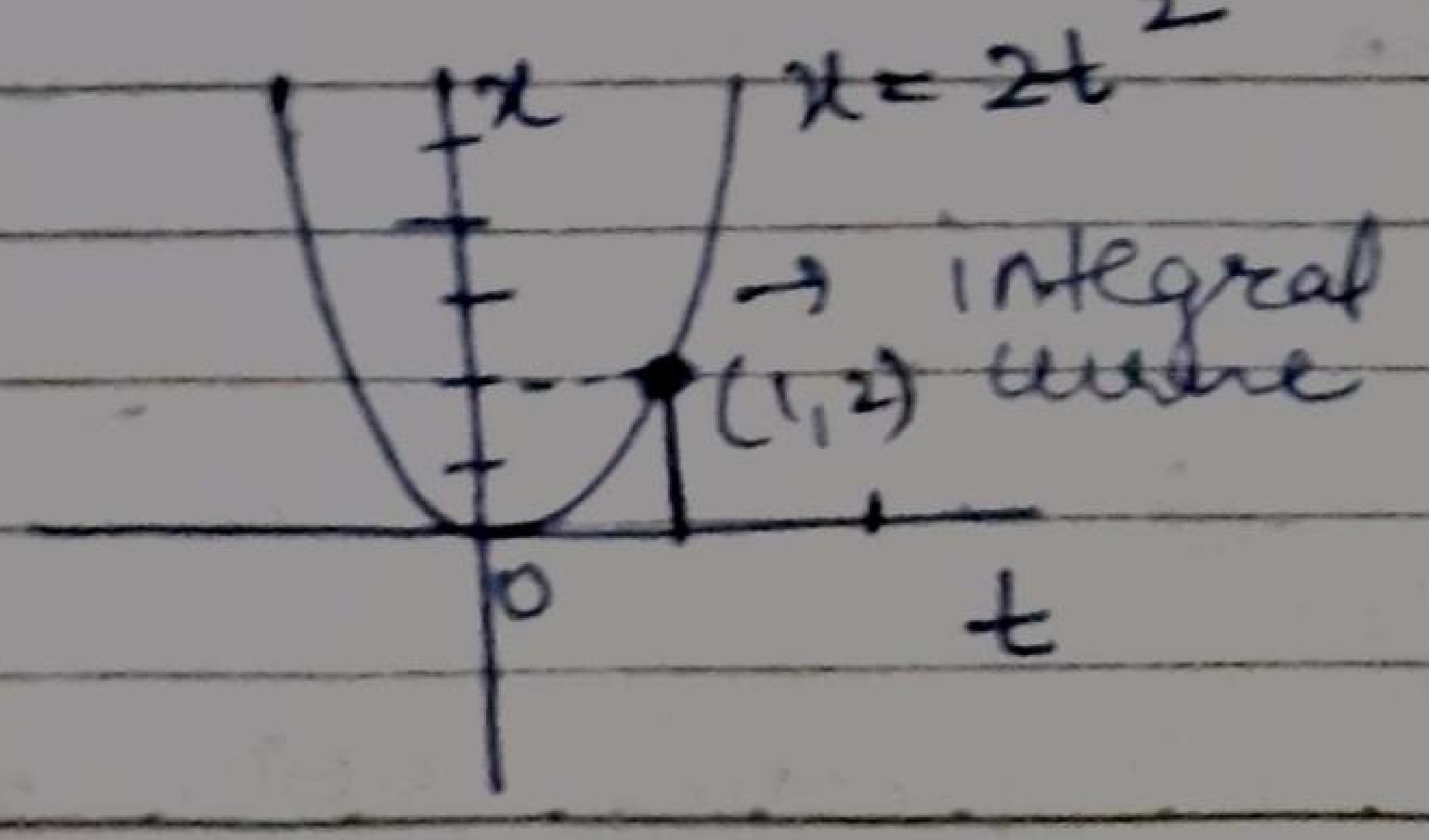
$t\dot{x} = 2x$ with $x(1) = 2$

$x(t) = ct^2$

$x(1) = c = 2$

So, $x(t) = 2t^2$ is the desired solution

graph



Sec 21.5

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS I

A first order linear differential equation can be written in the form:

$$\dot{x} + a(t)x = b(t)$$

where a and b are continuous functions of t in a certain interval and $x = x(t)$ is the unknown function. The functions $a(t)$ and $b(t)$ may very well represent expressions such as t^2 and e^t or may be constants also.

Examples: ① $\dot{x} + x = t$

② $\dot{x} + 2x = 6$

③ $\frac{dy}{dt} + 3t^2 y = 0$

④ $2\frac{dy}{dt} + 12y + 2e^t = 0$

⑤ $\dot{x} + \frac{e^t}{(t^2+1)} x = \frac{t \ln t}{(t^2+1)}$

Solution of $\dot{x} + a(t)\, x = b(t)$
where $a(t)$ is a constant $= a$

i.e. $\dot{x} + ax = b(t)$

Let $b(t) = 0$, then
$\dot{x} = -ax$

General sol^n $\frac{dx}{dt} = -ax$

$$\int \frac{1}{x}\, dx = \int -a\, dt$$

$$\ln x = -at + \ln A$$
$$\ln x = \ln e^{-at} + \ln A$$
$$x(t) = A e^{-at} \rightarrow \text{Gen sol}^n$$

$t = 0$, $x(0) = A e^{-0} = A$

Or
$$x(t) = x(0)\, e^{-at} \rightarrow \text{Particular Sol}^n$$

Now if $b(t) \neq 0$

$$\dot{x}(t) + a\, x = b(t) \quad (1)$$

Let us multiply by e^{+at}

$$e^{+at}\left[\dot{x}(t) + a\, x(t)\right] = e^{+at}\, b(t)$$

Consider the derivative of
$e^{+at} x(t)$

$$\frac{d}{dt}\left(e^{+at} x(t) \right) = e^{+at} \dot{x} + x e^{+at} (+a)$$

so (1) can be rewritten as:

$$\frac{d}{dt}\left(e^{at} x(t) \right) = e^{at} b(t)$$

So for some c,

$$e^{at} x(t) = \int e^{at} b(t) dt + C$$

Multiply by e^{-at}, we get

$$x(t) = C e^{-at} + e^{-at} \int e^{at} b(t) dt$$

[e^{at} is called the INTEGRATING FACTOR enables us to solve the differential equation in (1)]

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$\therefore$ Keep in mind

$$\boxed{\dot{x} + ax = b(t) \Leftrightarrow x(t) = e^{-at}\left[C + \int e^{at} b(t)\,dt\right]}$$

where C is a constant

In particular, if $b(t) = b$, then

$$\int e^{at}\, b\, dt = b \int e^{at}\, dt = \left(\frac{b}{a}\right) e^{at}$$

and so —:

$$\boxed{\dot{x} + ax = b \Leftrightarrow x(t) = Ce^{-at} + \frac{b}{a}}$$ where C is a constant

EQUILIBRIUM STATE AND TEST FOR STABILITY

Equilibrium is defined as a state of rest and is achieved when $\dot{x} = \frac{dx}{dt} = 0$.

when $\dot{x} + ax = b$

At Eq^n: $\dot{x} = 0 \Rightarrow x = b/a$

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An equilibrium is either stable or unstable. An equilibrium is a stable equilibrium if there is a tendency towards it if we are not at equilibrium.

(and $a>0$)

If for $\dot{x} + ax = b$, eq^n is stable because

$a > 0$ and $t \to \infty$, then

$$x(t) = Ce^{-at} + \frac{b}{a} \to b/a$$

However if,

$a<0$ then eq^n is unstable because $x(t)$ will be non-convergent as sol^n does not tend to a finite value as $t \to \infty$.

QUESTIONS

Find the general of the equation

$$\dot{x} + 2tx = t$$

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QUESTIONS:

Q1 For the following, solve the differential equation and determine whether equation is stable.

(1) $\frac{dy}{dt} + 2y = 6$

(2) $\frac{dy}{dt} + 4y = 12$

(3) $2\dot{x} + 4x = 6$

Solⁿ (1) $y(t) = Ce^{-at} + e^{-at}\int e^{a(t)} b(t)dt$

$y(t) = Ce^{-2t} + e^{-2t}\int e^{2t}(6)\,dt$

$y(t) = Ce^{-2t} + 6e^{-2t}\int e^{2t}dt$

$= Ce^{-2t} + 6e^{-2t}\left(\frac{e^{2t}}{2}\right)$

$y(t) = Ce^{-2t} + 3$

(2) $\cancel{\frac{dy}{}}$

$y(t) = Ce^{-4t} + e^{-4t}\int e^{4t}(12)\,dt$

$= Ce^{-4t} + e^{-4t}(12)\int e^{4t}\,dt$

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$$= Ce^{-4t} + 12e^{-4t}\frac{e^{4t}}{4}$$

$$= Ce^{-4t} + 3$$

(3) $$2\dot{x} + 4x = 6$$

$$\dot{x} + 2x = 3$$

$$x(t) = Ce^{-2t} + e^{-2t}\left[\int e^{2t}(3)\,dt\right]$$

$$= Ce^{-2t} + 3e^{-2t}\int e^{2t}\,dt$$

$$= Ce^{-2t} + 3e^{-2t}\frac{e^{2t}}{2}$$

$$= Ce^{-2t} + 1.5$$

EQUILIBRIUM

(1) $$\frac{dy}{dt} = 0 \Rightarrow 2y = 6$$

$$\Rightarrow y = 3$$

$a = 2 > 0 \Rightarrow eq^n$ is stable

Neelgagan

(2) $\frac{dy}{dt} = 0 \Rightarrow y = 3$

$a > 0 \; (= 4)$ eqn is stable

(3) $a = 2 > 0$ eqn is stable

$\frac{dx}{dt} = \dot{x} = 0 \Rightarrow x = 3/2 = 1.5$

H&S

Q6. Let $N = N(t)$ denote the size of a certain population $X = X(t)$ the total production and $x(t) = \frac{X(t)}{N(t)}$ the production per capita at time t. T. Haavelmo studied the model described by the equations:

$$[1] \; \frac{\dot{N}}{N} = \alpha - \beta \frac{N}{X} \quad [2] \; X = AN^{\alpha}$$

where α, β, and a are positive constants with $a \neq 1$. Show that this leads to a differential equation of the form $\dot{x} + ax = b$ for $x = x(t)$.

Solve this equation and then find expressions for $N = N(t)$ and $X = X(t)$. Examine the limits for $x(t)$, $N(t)$ and $X(t)$ as $t \to \infty$ in the case $0 < a < 1$.

Sol^n

$$x = \frac{X}{N}$$

$$\ln x = \ln X - \ln N$$

Diff. w.r.t t, we get

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{X}\frac{dX}{dt} - \frac{1}{N}\frac{dN}{dt}$$

$$\frac{\dot{x}}{x} = \frac{\dot{X}}{X} - \frac{\dot{N}}{N} \quad \text{--- (1)}$$

Given $X = A N^{a}$

$$\dot{X} = A\alpha N^{a-1} \dot{N}$$

$$\frac{\dot{X}}{X} = \frac{A\alpha N^{a-1}\dot{N}}{X}$$

$$= \frac{\cancel{A}\alpha N^{a-1}\dot{N}}{\cancel{A} N^{a}}$$

$$\frac{\dot{X}}{X} = a\frac{\dot{N}}{N} \quad \text{--- (2)}$$

Substitute ② in ①, we get

$$\frac{\dot{x}}{x} = a\frac{\dot{N}}{N} - \frac{\dot{N}}{N} = (a-1)\frac{\dot{N}}{N}$$

$$\frac{\dot{x}}{x} = (a-1)\left(\alpha - \frac{\beta}{x}\right)$$

$$\frac{\dot{x}}{\cancel{x}} = \frac{(a-1)(\alpha x - \beta)}{\cancel{x}}$$

$$\dot{x} = (a-1)\alpha x - (a-1)\beta$$

$$\dot{x} - \alpha(a-1)x = -(a-1)\beta.$$

Sol^n of this is : $x(t) = Ce^{-at} + b/a$
when $\dot{x} + ax = b$

$$x(t) = Ce^{\alpha(a-1)t} + \frac{+\cancel{(a-1)}\beta}{+\alpha\cancel{(a-1)}}$$

$$x(t) = Ce^{\alpha(a-1)t} + \beta/\alpha \quad \text{— ③}$$

$t = 0$, $x = x(0)$

or $x(0) = Ce^{\alpha(a-1).0} + \beta/\alpha$

$\Rightarrow C = (x(0) - \beta/\alpha)$ — ④

Plugging (4) in (3), we get

$$x(t) = \left(x(0) - \beta/\alpha\right) e^{\alpha(a-1)t} + \beta/\alpha$$

Now $X = A N^{a}$

$$x = \frac{X}{N}$$

$\Rightarrow X = xN$

so $xN = A N^{a}$

$\Rightarrow x = A N^{a-1}$

$$N(t) = \left(\frac{x}{A}\right)^{\frac{1}{a-1}}$$

$$X(t) = xN = x\left(\frac{x}{A}\right)^{\frac{1}{a-1}}$$

or
$$X(t) = \frac{1}{(A)^{\frac{1}{a-1}}} x^{\frac{2-a}{1-a}}$$

Given $0 < a < 1$

Let $t \to \infty$, $x(t) \to \beta/\alpha$

So
$$N(t) \to \left(\frac{\beta/\alpha}{A}\right)^{\frac{1}{a-1}}$$

$$X(t) \to A\left(\frac{\beta/\alpha}{A}\right)^{\frac{a}{a-1}}$$

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FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Now, we shall study the solution of the general linear equation

$\dot{x} + a(t)x = b(t)$ where the coefficient of x i.e. $a(t)$ is not constant.

If $b(t) = 0$, then

$$\dot{x} + a(t)x = 0$$

$$\frac{dx}{dt} = -a(t)x$$

$$\int \frac{1}{x}\,dx = -\int a(t)\,dt$$

$$\ln x = -\int a(t)\,dt$$

$$x(t) = C e^{-\int a(t)dt}$$

If $b(t) \neq 0$, then

$$\dot{x} + a(t)x = b(t) \quad \text{— (1)}$$

Let us multiply by integrating factor: $e^{\int a(t)dt}$

$$e^{\int a(t)dt}\dot{x} + e^{\int a(t)dt}a(t)x = e^{\int a(t)dt}b(t)$$

$\hookrightarrow$ (2)

Now, consider the derivative

of $x e^{\int a(t)\,dt}$.

$$\frac{d}{dt}\left(x e^{\int a(t)dt}\right) = x e^{\int a(t)\,dt} a(t) + e^{\int a(t)\,dt} \dot{x}$$

which is LHS of (2)

So we can write :-

$$\frac{d}{dt}\left(x e^{\int a(t)dt}\right) = e^{\int a(t)\,dt} \cdot b(t)$$

Integrating both sides for some C we can write,

$$x e^{\int a(t)dt} = \int e^{\int a(t)\,dt} \cdot b(t)\,dt + C$$

Multiplying by $e^{-\int a(t)\,dt}$ we get

$$x(t) = e^{-\int a(t)\,dt}\left[C + \int e^{\int a(t)\,dt} \cdot b(t)\,dt\right]$$

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QUESTIONS

Q Find the general solution of the equation: $\frac{dy}{dt} + 2ty = t$

Ans $a(t) = 2t$, $b(t) = t$

Solⁿ is : $y(t) = e^{-\int a(t)\,dt}\left[c + \int (e^{\int a(t)\,dt} \cdot b(t))\,dt\right]$

$$y(t) = e^{-\int 2t\,dt}\left[c + \int (e^{\int 2t\,dt} (t))\,dt\right]$$

$$= e^{-(t^2+k)}\left[c + \int t\, e^{t^2+k} \cdot dt\right]$$

$$= Ce^{-(t^2+k)} + e^{-(t^2+k)}\left[\int t e^{t^2+k}\,dt\right]$$

$$= Ce^{-(t^2+k)} + e^{-t^2}\int t\, e^{t^2}\,dt$$

$$= Ce^{-(t^2+k)} + e^{-t^2}\left(\frac{1}{2}e^{t^2} + m\right)$$

Note: $\int t\, e^{t^2}\,dt$

Let $e^{t^2} = l$

$e^{t^2}\, 2t\,dt = dl$

$$\int e^{t^2} \cdot t\,dt = \int \frac{1}{2}\,dl = \frac{1}{2} l + m = \frac{1}{2} e^{t^2} + m$$

or $y(t) = Ce^{-(t^2+k)} + \frac{1}{2} + e^{-t^2} \cdot m$

$y(t) = e^{-t^2}\left[Ce^{-k} + m\right] + \frac{1}{2}$

Q $\frac{dy}{dt} + t^2 y = 5t^2$

$a(t) = t^2 \quad , \quad b(t) = 5t^2$

Sol^n

$y(t) = e^{-\int a(t)dt}\left[C + \int e^{a(t)\,dt}\, b(t)\, dt\right]$

$\int a(t)\,dt = \int t^2 dt = \frac{t^3}{3} + k$

$\to y(t) = e^{-\left(\frac{t^3}{3}+k\right)}\left[C + \int e^{\left(\frac{t^3}{3}+k\right)} \cdot (5t^2)\,dt\right]$

$y(t) = e^{-\left(\frac{t^3}{3}+k\right)} \cdot C + 5e^{-\left(\frac{t^3}{3}+k\right)} \int e^{\frac{t^3}{3}+k} \cdot t^2\, dt$

$= Ce^{-\left(\frac{t^3}{3}+k\right)} + 5e^{-t^3/3} \int e^{t^3/3} \cdot t^2\, dt$

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Let $e^{t^3/3} = l$

$$e^{t^3/3} \cdot \frac{3t^2}{3}\, dt = dl$$

$$\int \left(e^{t^3/3} \cdot t^2\right) dt = \int dl$$

$$\int \left(e^{t^3/3} \cdot t^2\right) dt = l + m$$

$$= e^{t^3/3} + m$$

so

$$y(t) = C\, e^{-(t^3/3 + k)} + 5\, e^{-t^3/3} \left[e^{t^3/3} + m \right]$$

$$= C\, e^{-t^3/3}\, e^{-k} + 5 + 5m\, e^{-t^3/3}$$

$$= e^{-t^3/3} \left[C e^{-k} + 5m \right] + 5$$

Solution of a General linear equation when $x(t_0) = x_0$ is given

Suppose at $t = t_0$, $x(t_0) = x_0$ is known.

Solution to $\dot{x} + a(t)x = b(t)$ is given by:

$$x(t) = e^{-\int a(t)dt}\left[C + \int e^{\int a(t)dt} b(t)dt\right]$$

Let $F(t)$ be a function such that $\dot{F}(t) = b(t)e^{A(t)}$ where $A(t) = \int a(t)\,dt$ and so

$$x(t) = C e^{-A(t)} + e^{-A(t)} F(t)$$

For $t = t_0$, we get

$$x(t_0) = C e^{-A(t_0)} + e^{-A(t_0)} F(t_0)$$

$$\Rightarrow \boxed{C = x(t_0)e^{A(t_0)} - F(t_0)}$$

Substituting value of c in $x(t)$ we get

$$x(t) = x(t_0)\, e^{-[A(t)-A(t_0)]} + e^{-A(t)}\left[F(t)-F(t_0)\right]$$

From the definition of $F(t)$ we get

$$F(t) - F(t_0) = \int_{t_0}^{t} b(s)\, e^{A(s)}\, ds$$

$$e^{-A(t)}\left[F(t) - F(t_0)\right] = e^{-A(t)} \int_{t_0}^{t} b(s)\, e^{A(s)}\, ds$$

$$= \int_{t_0}^{t} b(s)\, e^{-(A(t)-A(s))}\, ds$$

$\therefore$, we have the following result:

$$\dot{x} + a(t)x = b(t) \iff x = x_0\, e^{-\int_{t_0}^{t} a(k)\,dk} + \int_{t_0}^{t} \left(b(s)\, e^{-\int_s^t a(k)\,dk}\right) ds$$

Qualitative Theory and Stability

In the previous sections, we solved the differential equations explicitly. But many a times economic phenomenon is described by differential equations which cannot be solved explicitly in terms of elementary function. The differential equations may also contain unspecified functions or parameters. Under this scenario we would like to find qualitative solutions and analyse stability of the equilibrium, if any.

EQUILIBRIUM, STABILITY & PHASE DIAGRAM:

Equilibrium which is also called a state of rest is a state when once achieved, there is no tendency to move away from it.

Suppose we have the differential

equation $\dot{x} = F(x)$ which is a special case of differential equation $\dot{x} = F(t, x)$ where t does not explicitly appear on the right hand side. This is called an AUTONOMOUS differential equation. This differential equation shall be examined for presence of equilibrium and its stability.

EQUILIBRIUM/STATIONARY STATE.

Defined by $\dot{x} = 0$ or $F(a) = 0$
So $x(t) = a \ \forall t$.

STABILITY ANALYSIS

To examine the stability of the equilibrium, it is necessary to study phase diagrams. Phase Diagram, is the graph of $y = \dot{x} = F(x)$ drawn on $x\dot{x}$ plane together with the arrows of motion. Phase diagram helps us to determine the qualitative properties of a solution to a differential equation w/o solving it.

if we are away from it which implies that it is stable equilibrium state. Please note that $F'(a_2) < 0$. So, we get the following result:

(a) $F(a) = 0$ and $F'(a) < 0 \Rightarrow$ a is a stable equilibrium state for $\dot{x} = F(x)$.

(b) $F(a) = 0$ and $F'(a) > 0 \Rightarrow$ a is an unstable equilibrium state for $\dot{x} = F(x)$.

NOTE:

$$\dot{x} = F(x) = x - x^2$$

$$F'(x) = \frac{d\dot{x}}{dx} = 1 - 2x$$

$$F'(a_1) = F'(0) = 1 > 0$$

$$F'(a_2) = F'(1) = -1 < 0$$

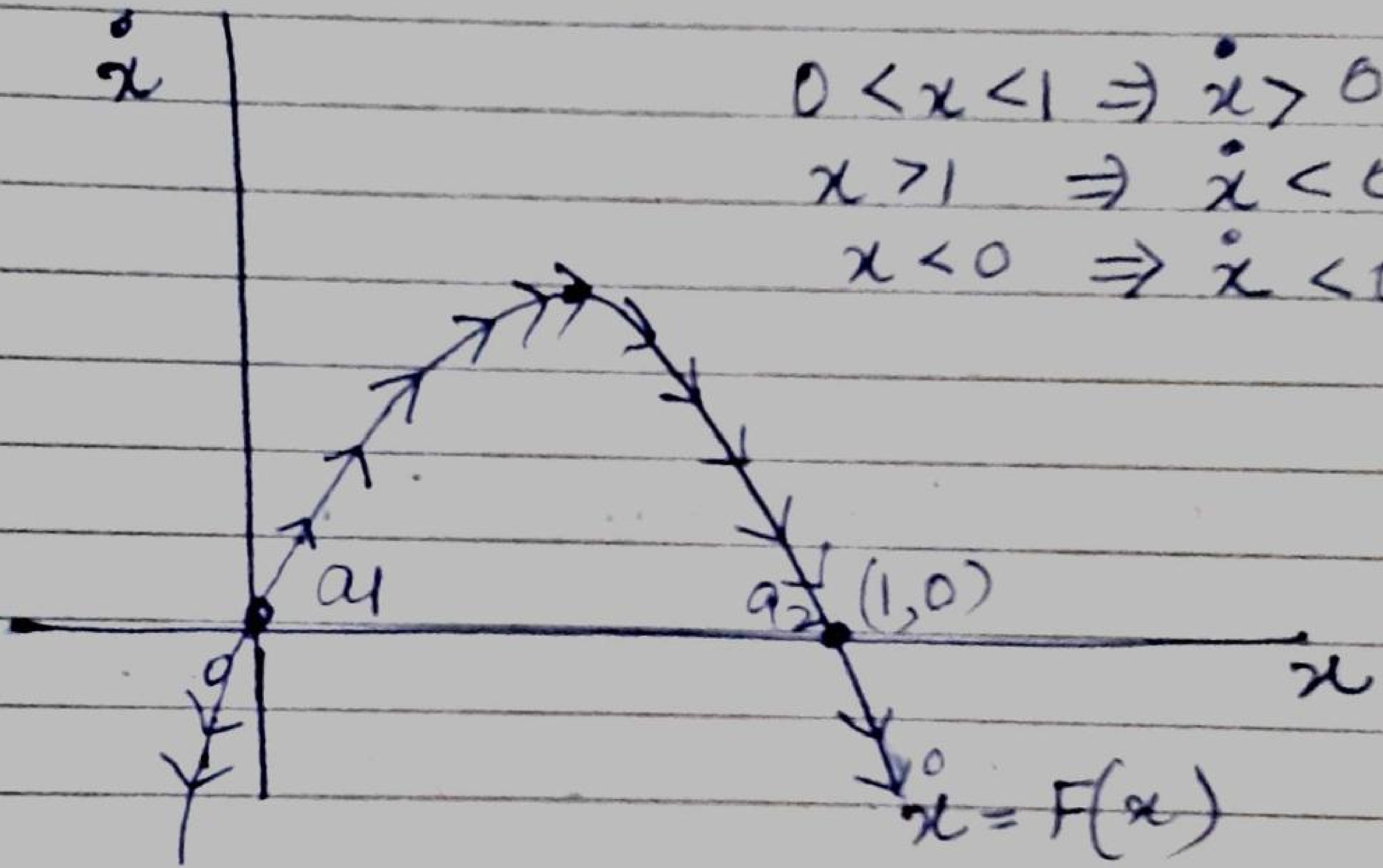
Arrows of motion shows the movement of x as t increase if we are at any point on the phase line. If we are at a point on the curve above the x axis then $F(x(t)) > 0$ and so $\dot{x} = F(x(t)) > 0$, so that $x(t)$ increases with t. It follows that the point $(x, \dot{x})$ move from left to the right in the diagram if we are above the x-axis. If, on the other hand, we are at a point on the graph below the x axis, then $\dot{x}(t) < 0$ and $x(t)$ decreases with t, so we move from right to left.

Consider $a_1 = 0$; Arrow of motion indicate that a_1 is unstable. Also note that $F'(a_1) > 0$. Because if we are away from it, overtime we will not move towards it. However, at $a_2 = 1$, arrows of motion indicate a movement towards it

Consider an example :

$$\dot{x} = F(x) = x - x^2$$
$$\dot{x} = x(1-x)$$

$0 < x < 1 \Rightarrow \dot{x} > 0$
$x > 1 \Rightarrow \dot{x} < 0$
$x < 0 \Rightarrow \dot{x} < 0$



The plot of $\dot{x} = F(x)$ in $x\dot{x}$ plane is called <u>Phase Line</u>.

$\dot{x} = 0 \Rightarrow x = 0$ or $x = 1$
are the equilibrium state.
Because when $x = 0 \,/\, x = 1$
$F(0) = \dot{x} = 0 \;/\; F(1) = \dot{x} = 0$

$\therefore$ once these x values are achieved $x = 0/1$
$\forall\, t$ afterwards.

NEOCLASSICAL GROWTH THEORY (ROBERT SOLOW)

Solow growth model was proposed by Nobel laureate Prof. Robert Solow. Solow analysed the long term path of labour, capital and output by analysing the case where capital and labour can be combined in varying proportions. Solow's production function appear as:

$$Y = F(K, L) \qquad (K, L > 0)$$

where Y is output (net of depreciation), K is capital and L is labour. It is assumed that F_K and F_L are positive (MP_K and $MPL > 0$) and F_{KK} and F_{LL} are negative (diminishing returns to each input).
Production function F is taken to be linearly homogeneous so $Y = L\,F(K/L, 1) = L\,f(k)$ where $k = K/L$.

$$F_K = MP_K = f'(k)$$

Since $MP_K > 0 \Rightarrow f'(k) > 0$

$$F_{KK} = \frac{\partial f'(k)}{\partial K} = \frac{d f'(k)}{dk} \cdot \frac{dk}{dK}$$

$$= f''(k) \cdot \frac{1}{L}$$

So $F_{KK} < 0 \Rightarrow f''(k) < 0$

$\therefore$ $f(k)$ is a function which increases with k at a decreasing rate.

Solow's assumptions are:

$$\dot{K} \left(\frac{dK}{dt}\right) = I = sY \quad (\text{constant saving rate})$$

$$\frac{\dot{L}}{L} = \left(\frac{dL/dt}{L}\right) = n > 0$$

Labour force grows exponentially at the rate n.

Substituting Y $(= Lf(k))$ into $\dot{K} = sLf(k)$

Now $k = K/L$

So $\dot{K} = \frac{dK}{dt} = k\frac{dL}{dt} + L\frac{dk}{dt}$

$$\Rightarrow \frac{dk}{dt} = \frac{\dot{K}}{L} - \frac{k}{L}\frac{dL}{dt}$$

$$\boxed{\dot{k} = s f(k) - nk}$$

where k is capital per worker, s constant saving rate, f is prodn function and n is the rate of growth of labour

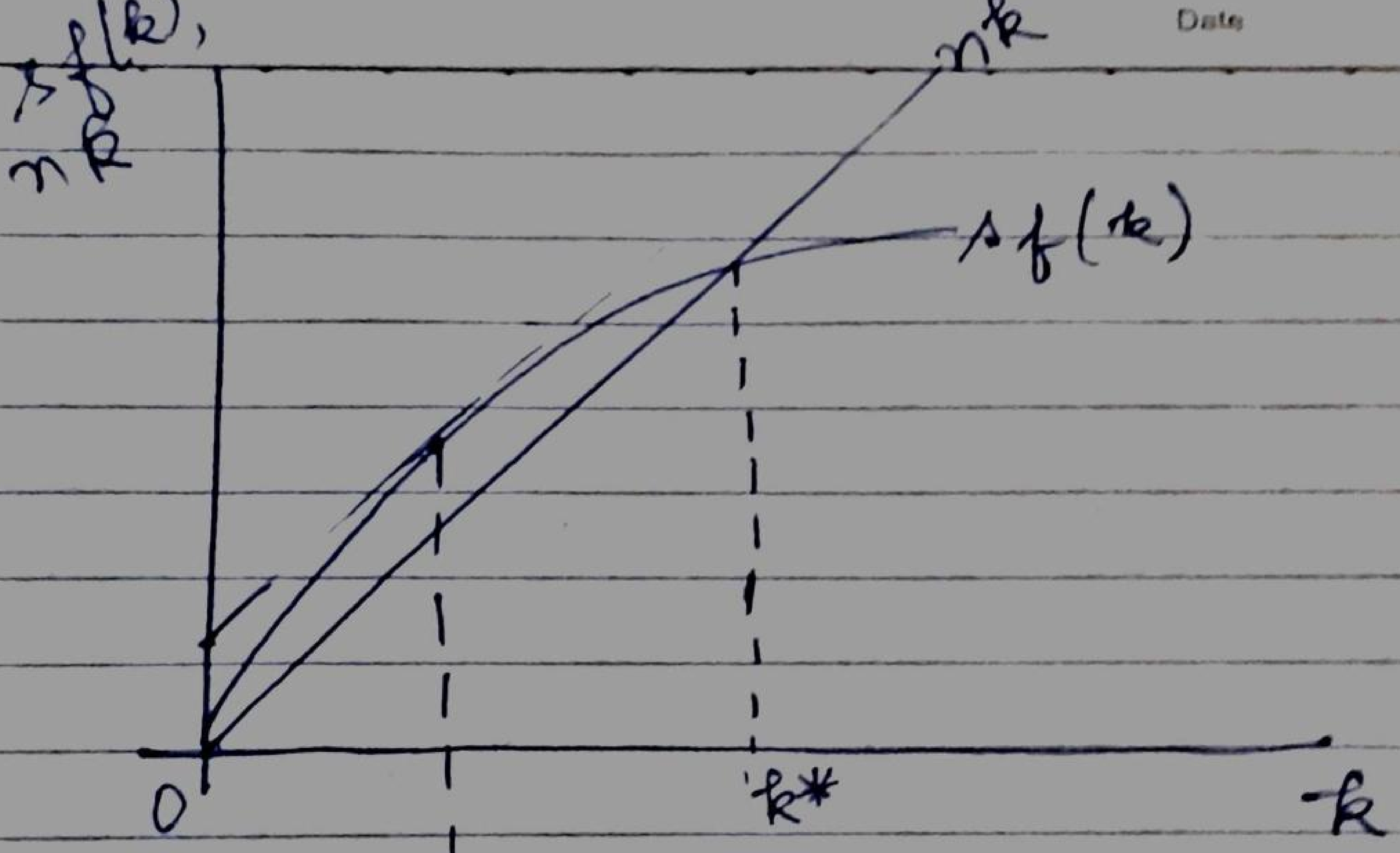
$\dot{k}$ describes the growth of economy in this model. Given the assumptions of solow model, the differential equation can be analyzed to find out the properties of the growth path of the economy. This equation is the fundamental equation of the solow growth model.

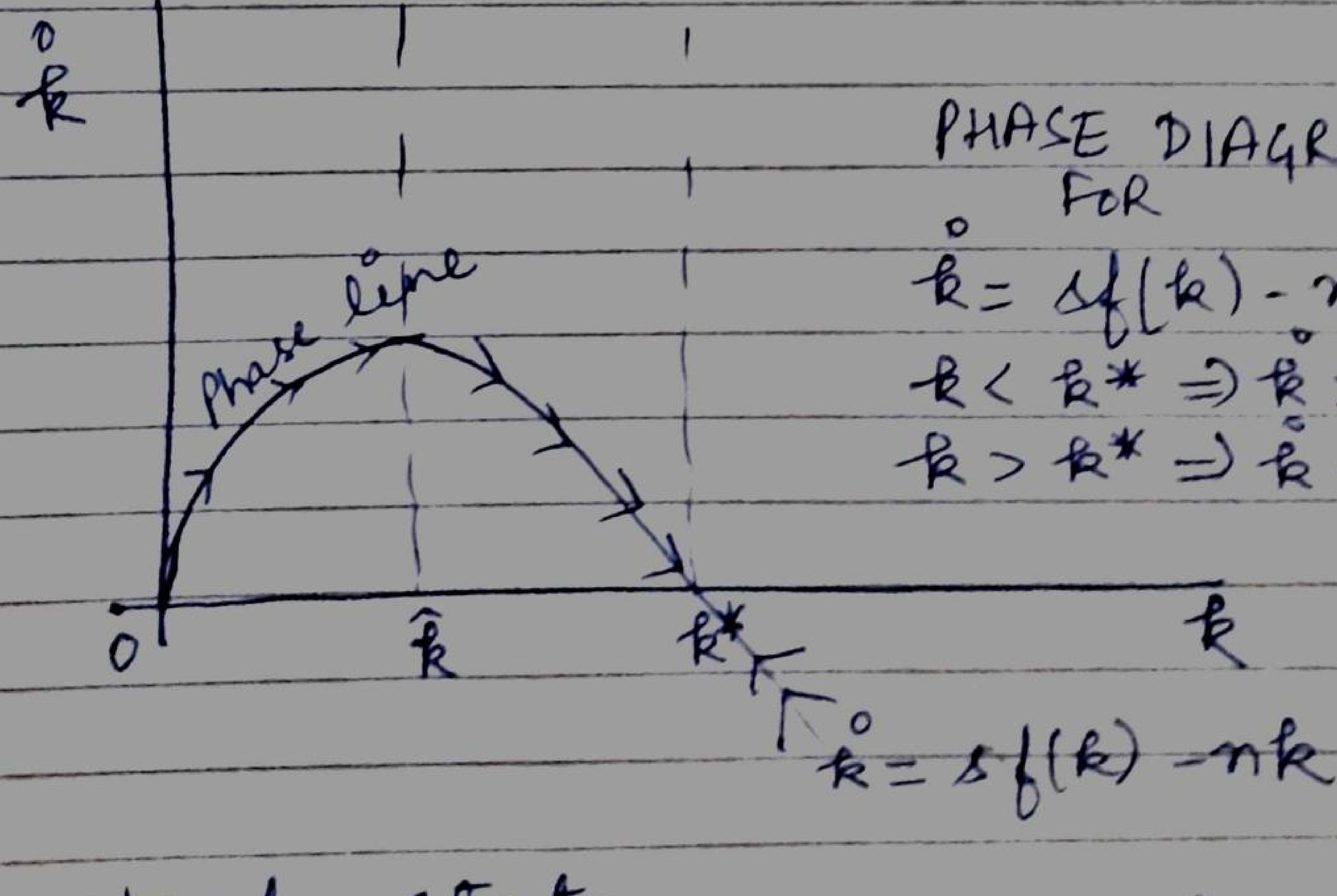
QUALITATIVE - GRAPHIC ANALYSIS

Since the differential equation $\dot{k} = s f(k) - nk$ is stated in a general function form, no specific quantitative solution is available. But, we can analyse the solution qualitatively. For this, we shall draw the phase diagram.

We know that, $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0 \ \forall k > 0$, $f'(k) \to \infty$ as $k \to 0$ and $f'(k) \to 0$ as $k \to \infty$.

Let's plot $s f(k)$ and nk as separate curves. nk is a linear function with 0 intercept and slope n. $s f(k)$ is a curve which increases at a decreasing rate. Based on the two curves, $\dot{k}$ is achieved as a vertical difference between the two curves.





PHASE DIAGRAM
FOR
$\mathring{k} = sf(k) - nk$

$k < k^* \Rightarrow \mathring{k} > 0$

$k > k^* \Rightarrow \mathring{k} < 0$

<u>Steady state</u>

It is achieved when $\mathring{k} = 0$

$\Rightarrow k = 0$ or $k = k^*$

k^* is stable equilibrium because the phase line is

$\left(\text{At } k^* \ \frac{d\dot{k}}{dk} = s f'(k) - n < 0 \Rightarrow \text{stability} \right)$

negatively sloped at k^*.
The significant point is that once this equilibrium is attained capital-labour ratio shall not vary overtime (steady state) i.e. capital will grow at the same rate as labour (i.e. n).
Also, in order to satisfy $Y = L f(k)$, the output will also grow at the same rate of L and K i.e n.
Such a situation, when all the relevant variables grow at an identical rate is called a STEADY STATE.

Note:-
Mathematical Proof:
Given $\frac{1}{L}\frac{dL}{dt} = n$

At $k^* \Rightarrow \dot{k} = 0 \Rightarrow \frac{d}{dt}(K/L) = 0$

$$\Rightarrow \frac{1}{K}\frac{dK}{dt} = \frac{1}{N}\frac{dL}{dt} = n$$

Now $Y = L f(k)$

$$\ln Y = \ln L + \ln f(k)$$

$$\Rightarrow \frac{1}{Y}\frac{dY}{dt} = \frac{1}{L}\frac{dL}{dt} + \frac{1}{f(k)} f'(k) \frac{dk}{dt}$$

$$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{L}\frac{dL}{dt} + 0 = n$$