Study Material(IV)

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1 COORDINATIZATION

If B is a basis for a vector space V, then we know that every vector in V has a unique expression as a linear combination of the vectors in B. For example, the vector $[a_1, a_2, ..., a_n]$ in $\mathbb{R}^n$ is written as linear combination of the standard basis $\{e_1, e_2, .., e_n\}$ of $\mathbb{R}^n$ in a unique way. Dealing with the standard basis in $\mathbb{R}^n$ is easy because the coefficients in the linear are the same as the coordinates of the vectors. However, this is not necessarily true for other bases.

In this section we develop a process called **Coordinatization** for representing any vector in a finite dimensional vector space in terms of its coefficients with respect to a given basis. We also learn how the coordinatization changes whenever we change bases.]

DEFINITION (Ordered Basis) :
An ordered basis for a vector space V is an ordered n-tuple of vectors $(v_1, v_2, ..., v_n)$ such that the set $\{v_1, v_2, ..., v_n\}$ is a basis for V.

Note : 1 In an ordered basis the elements are written in a specific order. Thus (i, j, k) and (j, i, k) are different ordered basis for $\mathbb{R}^n$.

DEFINITION (Coordinatiation of a Vector) :
Let $B = \{v_1, v_2, ..., v_n\}$ be an ordered basis for an $n-dimensional$ vector space V. Let v be a vector in V such that

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

Then the coordinate vector of v with respect to the ordered basis B, denoted by $[v]_B$ is defined as

$$[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

Thus the vector $[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$ is frequently called as "v expressed in $B-coordinates$.

Note:2 Notice that the coordinattization of any vector $v = [a_1, a_2, ..., a_n]$ in $\mathbb{R}^n$ with repect to the standard ordered basis $B = \{e_1, e_2, ..., e_n\}$ is

Thus the vector

$$[v]_B = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

Example 1 Consider the vector space $\mathbb{R}^n$ and let $B = \{v_1 = [1, 1, 0], v_2 = [2, 0, 1], v_3 = [0, 1, 2]\}$ be an ordered basis for $\mathbb{R}^n$. If $v = [1, 1, -5]$, compute $[v]_B$

Solution To find $[v]_B$ we need to find scalars a_1, a_2, a_3 such that

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$\Longrightarrow \ [v]_B = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\Longrightarrow$$

$$\left.\begin{array}{lll} a_1 & +2a_2 & +0a_3 = 1 \\ a_1 & +0a_2 & +a_3 = 1 \\ 0a_1 & +a_2 & +2a_3 = -5 \end{array}\right\} \qquad (1)$$

Now we solve this linear system using Row Reduced Echelon form. We get $a_1 = 3, a_2 = -1$ and $a_3 = -2$ and so

$$[v]_B = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Exercise 1. Try to solve Example 3 (from Andrilli page No. 282).

Mehtod for Coordianatizing a vector with respect to a finite ordered basis

Let V be a nontrivial vector subspace of $\mathbb{R}^n$, let $B = \{v_1, v_2, ..., v_k\}$ be an ordered basis for V, and let $v \in \mathbb{R}^n$. To calculate $[v]_B$, if possible perform the following steps:

Step 1: Form an augmented matrix $[A|v]$ by using the vector in B as the columns of A, in order and using v as a column on the right.

Step 2: Row reduce $[A|v]$ to obtain the reduced row echelon form $[C|w]$.

Step 3: If there is row of $[C|w]$ that contains all zeroes on the left and has a nonzero entry on the right, then $v \notin span(B) = V$. and coordinatization is not possible. Stop.

Step 4: Otherwise, $v \in span(B) = V$. Eliminate all rows consisting entirely of zeroes in $[C|w]$ to obtain $[I_k|y]$. Then, $[v]_B = y$, the last column of $[I_k|y]$.

Exercise 2. Let V be the subspace of $\mathbb{R}^n$ spanned by the ordered basis $B = \{[2, 0, 1], [1, 2, 0], [1, 1, 1]\}$. Find $[v]_B$, if possible for the vector $v = [4, -9, 5]$.

Fundamental Properties of Coordinatization

Theorem 1 *Let* $B = [v_1, v_2, ..., v_n]$ *be an ordered basis for a vector space V. Suppose* $w_1, w - 2, ..., w_k \in V$ *and* $a_1, a - 2, ..., a_k$ *are scalars. Then*

(a) $[w_1 + w_2]_B = [w_1]_B + [w_2]_B$

(b) $[a_1 w_1]_B = a_1[w_1]_B$

(c) $[a_1 w_1 + a_2 w_2 + ... + a_k w_k] = a_1[w_1]_B + a_2[w_2]_B + ... + a_k[w_k]_B$

2 THE TRANSITION MATRIX

In this section our goal is to determine how the coordinates of a vector change when we convert from one basis to another.

Definition

Suppose that V is a nontrivial $n - dimensional$ vector space with ordered bases B and C. Let P be the $n \times n$ matrix whose ith column, for $1 \neq i \neq n$, equals $[b_i]_C$, where b_i is the ith basis vector in B. Then P is called the transition matrix from $B - coordinates$ to $C - coordinates$.

We often say that P is the **"transition matrix from B to C."**

Note 3. To find the transition matrix P from the $B - basis$ to $C - basis$, we first compute the coordinate vector of each element of the $B - basis$ with respect to $c - basis$. we then form a matrix using these vectors as columns arranged in their natural order. The matrix so obtained is the transition matrix P.

Example 2. Consider the following ordered basis for $\mathbb{R}^3 : B = \{v_1 = [1, 2, 3], v_2 = [-2, 1, 0], v_3 = [1, 0, 1]\}$ and $C = \{w_1 = [1, 1, 0], w_2 = [0, 1, 2], w_3 = [1, 1, 1]\}$ find the transition matrix P from C to B.

Solution To find the transition matrix $P_{B \leftarrow C}$, we must find the coordinates of each vector in C with respect to the ordered basis B. we applying the coordinatization method to w_1, w_2 and w_3 simultaneously to obtain coordinates of each vector w_1, w_2 and w_3. Thus we have augmented matrix is

$$[v_1 v - 2v_3 | w_1 w_2 w_3]$$

Thus we Row Reduced the augmented matrix

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & -\frac{9}{2} & 2 & -2 \end{array}\right]$$

This gives

$$[w_1]_B = \begin{bmatrix} \frac{3}{2} \\ -2 \\ -\frac{9}{2} \end{bmatrix}, [w_2]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, [w_3]_B = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

These three vectors form the column of the transition matrix from C to B.

Note 4. Notice that the transition matrix P from $C - basis$ to $B - basis$ satifies the equation

$$AP = D$$

where A is the matrix whoes column are the vectors in B and D is the matrix whose columns are the vectors in C. In fact, this result is always true for any two ordered bases B and C of $\mathbb{R}^n$

Example 3. Consider the ordered basis $B = \{[1,0,1],[1,1,0],[0,0,1]\}$ for $\mathbb{R}^3$. Find another ordered basis C for $\mathbb{R}^3$ such that the transition matrix from C to B is:

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

solution Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ be the matrix whose column are vectors in B. Let D be the matrix whose columns are vectors in C. Then according to Note, D is given by

$$D = AP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

thus, the ordered basis $C = \{[3,2,0],[2,1,0],[3,1,3]\}$

Exercise 3. Let $B = \{[1,0,-1],[10,5,4],[2,1,1]\}$ and $C = \{[1,0,2],[5,2,5],[2,1,2]\}$ be ordered bases for $\mathbb{R}^3$. Find the transition matrix $P_{C \leftarrow B}$ from the $B - bases$ to $C - bases$.

Method for Calculating a Transition Matrix (Transition Matrix Method) Suppose that V is a non-trivial $k - dimensional$ subspace of $\mathbb{R}^n$ with ordered bases $B = \{v_1, v_2, ..., v_n\}$ and $C = \{w_1, w_2, ..., w_n\}$. Then to find the transition matrix P from C to B, row reduce the augmented matrix $[v_1, v_2, ..., v_k | w_1, w_2, ..., w_k]$ to obtain $\left[\begin{array}{c|c} I_k & P \\ rows & of zeroes \end{array}\right]$

Exercise 4. Try to solve Example 8 from Andrilli, Page No. 288.

Properties of Transition Matrix

Theorem 2 *Suppose that B and C are ordered basis for a non-trivial $n-$ dimensional vector space V, and let $v \in V$. Then the following properties of transition matrix holds:*

1. The trasition matrix $P_{B \leftarrow C}$ from C to B satisfies $P_{B \leftarrow C}[v]_C = [v]_B$.
2. The transition matrix $P_{B \leftarrow c}$ from C to B is invertible and $(P_{B \leftarrow C})^{-1} = P_{C \leftarrow B}$.

Exercise 5. Let $B = \{[1,0],[1,-3]\}$ and $C = \{[1,-1],[1,1]\}$ be two bases for $\mathbb{R}^2$. Let $v = [5,1]$

(a) Find the coordinate vector $[v]_C$ of v with respect to the $C - basis$.

(b) Find the coordinate vector $[v]_B$ of v with respect to the $B - basis$, using

$P_{B \leftarrow C}$.

(c) Compute the transition matrix $Q_{C \leftarrow B}$ from $B - basis$ to the $C - basis$ directly and verify that $Q_{C \leftarrow B} = (P_{B \leftarrow C})^{-}1)$.

Exercise 6. Consider a basis $B = \{b_1, b_2\}$ for $\mathbb{R}^2$, where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose that an v in $\mathbb{R}^2$ has the coordinate vector $[v]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find v.