Relativistic Kinematics

In this chapter I summarize the basic principles, notation, and terminology of relativistic kinematics. This is material you must know cold in order to understand Chapters 6 through 11 (it is not needed for Chapters *4* and *5*, however, and *if* you prefer you can read them *first).* Although the treatment is reasonably self-contained, I do assume that you have encountered special relativity before—if not, you should pause here and read the appropriate chapter in any introductory physics text before proceeding. *If you* are already quite familiar with relativity, this chapter *will* be an easy review—but read through it anyway because some of the notation may be new to you.

3.1 LORENTZ TRANSFORMATIONS

According to the special theory of relativity,' the laws of physics apply just as well in a reference system moving at constant velocity as they do in one at rest. An embarrassing implication of this is that there's no way of telling which system (if any) is at rest, and hence there is no way of knowing what "the" velocity of any other system might be. So perhaps I had better start over. Ahem.

According to the special theory of relativity,' the laws of physics are equally valid in all inertial reference systems. An inertial system is one in which Newton's first law (the law of inertia) is obeyed: objects keep moving in straight lines at constant speeds unless acted upon by some force.* It's easy to see that any two inertial systems must be moving at constant velocity with respect to one another, and conversely, that any system moving at constant velocity with respect to an inertial system **is** itself inertial.

* If you are wondering whether a freely falling system in a uniform gravitational field is "inertial," you know more than is good for you. Let's just **keep** gravity out of it.

Imagine, then, that we have two inertial frames, S and S', with S' moving at uniform velocity $\mathbf{v}$ (magnitude v) with respect to S (S, then, is moving at velocity $-\mathbf{v}$ with respect to S'). We may as well lay out our coordinates in such a way that the motion is along the common x/x' axis (Fig. 3.1), and set the master clocks at the origin in each system so that both read zero at the instant the two coincide (that is, $t = t' = 0$ when $x = x' = 0$). Suppose, now, that some event occurs at position (x, y, z) and time t in S. Question: What are the spacetime coordinates (x', y', z') and t' of this same event in S'? The answer is provided by the Lorentz transformations:

$$\begin{aligned} &\text{i.} \quad x' = \gamma(x - vt) \\ &\text{ii.} \quad y' = y \\ &\text{iii.} \quad z' = z \\ &\text{iv.} \quad t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned} \tag{3.1}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3.2}$$

The inverse transformations, which take us back from S' to S, are obtained by simply changing the sign of v (see Problem 3.1):

$$\begin{aligned} &\text{i}'. \quad x = \gamma(x' + vt') \\ &\text{ii}'. \quad y = y' \\ &\text{iii}'. \quad z = z' \\ &\text{iv}'. \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned} \tag{3.3}$$

The Lorentz transformations have a number of immediate consequences, of which I mention briefly the most important:

1. The relativity of simultaneity: If two events occur at the same time in S, but at different locations, then they do not occur at the same time in S'. Specifically, if $t_A = t_B$, then

$$t'_A = t'_B + \frac{\gamma v}{c^2}(x_B - x_A) \tag{3.4}$$

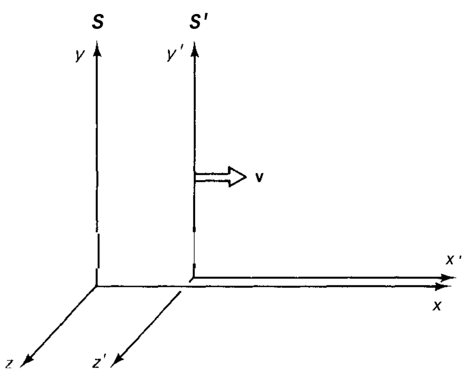

Figure 3.1 The inertial systems S and S'.

(see Problem 3.2). Events that are simultaneous in one inertial system, then, are not simultaneous in others.

2. Lorentz contraction: Suppose a stick lies on the x' axis, at rest in S'. Say one end is at the origin (x' = 0) and the other is at L' (so its length in S' is L'). What is its length as measured in ***S?*** Since the stick is moving with respect to *S,* we must be careful to record the positions of its two ends at the same instant, say t = 0. At that moment the left end is at x = 0 and the right end, according to equation (i), is at x = L'/γ. Thus the length of the stick is L = L'/γ, in S. Notice that γ is always greater than or equal to **1.** It follows that a moving object is shortened by a factor of y, as compared with its length in the system in which it **is** at rest. Notice that Lorentz contraction only applies to lengths along the direction *of* motion: perpendicular dimensions are not affected.

3. Time dilation: Suppose the clock at the origin in ***S'*** ticks off an interval T'; for simplicity, say it runs from t' = 0 to t' = T'. How long is this period as measured in S? Well, it begins at t = 0, and it ends when t' = T' at x' = 0, so [according to eq. **(iv')**] t = $\gamma T'$. Evidently the clocks in Stick off a longer interval, T = γ T', by that same factor of γ; or, put it the other way around: moving clocks run slow. Unlike Lorentz contraction, which is only indirectly relevant to elementary particle physics, time dilation is a commonplace in the laboratory. For in a sense every unstable particle has a built-in clock: whatever it is that tells the particle when its time is up. And these internal clocks **do** indeed run slow when the particle is moving. That is to say, a moving particle lasts longer (by a factor of y) than it would at rest.* (The tabulated lifetimes are, of course, for particles at rest.) In fact, the cosmic ray muons produced in the upper atmosphere would never make it to ground level were it not for time dilation (see Problem 3.4).

4. Velocity addition. Suppose a particle is moving in the x direction at speed u', with respect to S'. What is its speed, *u,* with respect to *S?* Well, it travels a distance A x = $\gamma(\Delta x' + v$ At') in a time At = $\gamma[\Delta t' + (v/c^2)\Delta x']$, so

$$\frac{\text{A x}}{\text{At}} = \frac{\Delta x' + v \text{ At'}}{\text{At'} + (v/c^2)\text{ Ax'}} = \frac{(\Delta x'/\Delta t') + v}{1 + (v/c^2)(\Delta x'/\Delta t')} .$$

But $\Delta x/\Delta t = u$, and $\Delta x'/\Delta t' = u'$, so

$$u = \frac{u' + v}{1 + (u'v/c^2)} \tag{3.5}$$

The numerator represents the classical answer to the same question, $u = u' + v$; the denominator introduces a relativistic correction that is small unless u' and v are close to c. Notice that if $u' = c$, then $u = c$ also: the speed of light is the same in all inertial systems.

* Actually, the disintegration of an individual particle **is** a random process; when we speak of a "lifetime" we really mean the average lifetime of that particle type. When I say that a moving particle lasts longer, I really mean that the average lifetime of a *group* of moving particles is longer.

3.2 FOUR-VECTORS

It is convenient at this point to introduce some simplifying notation. We define the position-time four-vector x^μ, $\mu = 0, 1, 2, 3$, as follows:

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z \tag{3.6}$$

In terms of x^μ, the Lorentz transformations take on a more symmetrical appearance:

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 - \beta x^1) \\ x^{1\prime} &= \gamma(x^1 - \beta x^0) \\ x^{2\prime} &= x^2 \\ x^{3\prime} &= x^3 \end{aligned} \tag{3.7}$$

where

$$\beta \equiv \frac{v}{c} \tag{3.8}$$

More compactly:

$$x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \qquad (\mu = 0, 1, 2, 3) \tag{3.9}$$

The coefficients Λ^μ_ν may be regarded as the elements of a matrix **Λ**:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ & & 0 & 0 \\ & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.10}$$

(i.e., $\Lambda^0_0 = \Lambda^1_1 = \gamma$; $\Lambda^1_0 = \Lambda^0_1 = -\gamma\beta$; $\Lambda^2_2 = \Lambda^3_3 = 1$; and all the rest are zero). To avoid writing lots of Σ's, we shall follow Einstein's "summation convention," which says that repeated Greek indices (one as subscript, one as superscript) are to be summed from 0 to **3.** Thus equation (3.9) becomes, finally,*

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu \tag{3.11}$$

A special virtue of this tidy notation is that the same form describes Lorentz transformations which are not along the x direction; in fact, the S and S' axes need not even be parallel; the **Λ** matrix is more complicated, naturally, but equation (3.11) still holds. [On the other hand, there is no real loss of generality in using expression (3.10), since we are always free to *choose* parallel axes, and to align the x axis along the direction of **v.]**

* In an expression such as this the Greek letter used for the summation index, ν, **is** of course completely arbitrary. The Same goes for the index μ, although it must match on the two sides of the equation. Thus equation (3.11) could just as well be written $x^{\kappa\prime} = \Lambda^\kappa_\lambda x^\lambda$. Either expression stands for the set of four equations:

$$\begin{aligned} x^{0\prime} &= \Lambda^0_0 x^0 + \Lambda^0_1 x^1 + \Lambda^0_2 x^2 + \Lambda^0_3 x^3 \\ x^{1\prime} &= \Lambda^1_0 x^0 + \Lambda^1_1 x^1 + \Lambda^1_2 x^2 + \Lambda^1_3 x^3 \\ x^{2\prime} &= \Lambda^2_0 x^0 + \Lambda^2_1 x^1 + \Lambda^2_2 x^2 + \Lambda^2_3 x^3 \\ x^{3\prime} &= \Lambda^3_0 x^0 + \Lambda^3_1 x^1 + \Lambda^3_2 x^2 + \Lambda^3_3 x^3 \end{aligned}$$

Although the individual coordinates of an event change, in accordance with equation **(3.11)**, when we go from S to S', there is a particular ***combination*** of them that remains the same (Problem **3.7**):

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 \quad \textbf{(3.12)}$$

Such a quantity, which has the same value in any inertial system, is called an ***invariant.*** (In the same sense, the quantity $r^2 = x^2 + y^2 + z^2$ is invariant under ***rotations.)*** Now, I would ***like*** to write this invariant in the form of a sum: $\sum_{\mu=0}^{3} x^\mu x^\mu$, but unfortunately there are those three imtating minus signs. To keep track of them, we introduce the ***metric,*** $g_{\mu\nu}$, whose components can be displayed as a matrix **g**:

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \textbf{(3.13)}$$

(i.e., $g_{00} = 1$; $g_{11} = g_{22} = g_{33} = -1$; all the rest are zero).* With the help of $g_{\mu\nu}$, the invariant I can be written as a double sum:

$$I = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x^\mu x^\nu \quad \textbf{(3.14)}$$

Carrying things a step further, we define the ***covariant*** four-vector x_μ (index down) as follows:

$$x_\mu \equiv g_{\mu\nu} x^\nu \quad \textbf{(3.15)}$$

(i.e., $x_0 = x^0$, $x_1 = -x^1$, $x_2 = -x^2$, $x_3 = -x^3$). To emphasize the distinction we call the "original" four-vector x^μ (index ***up)*** a ***contravariant*** four-vector. The invariant I can then be written in its cleanest form:

$$I = x_\mu x^\mu \quad \textbf{(3.16)}$$

All this will no doubt seem like monstrous notational overkill, just to keep track of three minus signs, but it's actually very simple, once you get used to it. (What's more, it generalizes nicely to non-Cartesian coordinate systems and to the curved spaces encountered in general relativity, though neither of these is relevant to us here.)

The position-time four-vector x^μ is the archetype for all four-vectors. We define a four-vector, a^μ, as a four-component object that transforms in the same way x^μ does when we go from one inertial system to another, to wit:

$$a^{\mu\prime} = \Lambda^\mu_\nu a^\nu \quad \textbf{(3.17)}$$

with the same coefficients Λ^μ_ν. To each such (contravariant) four-vector we as-

* I should warn you that some physicists define the metric with the opposite signs (−1, 1, 1, 1). It doesn't ***matter*** much—if I is invariant, so too is $-I$. But it does mean you must be on the lookout for unfamiliar signs. Fortunately, most ***particle*** physicists nowadays use the convention in equation (3.13).

sociate a covariant four-vector a_μ, obtained by simply changing the signs of the spatial components, or, more formally

$$a_\mu = g_{\mu\nu} a^\nu \tag{3.18}$$

Of course, we can go back from covariant to contravariant by reversing the signs again:

$$a^\mu = g^{\mu\nu} a_\nu \tag{3.19}$$

where $g^{\mu\nu}$ are technically the elements in the matrix $\mathbf{g}^{-1}$ (however, since our metric is its own inverse, $g^{\mu\nu}$ is the same as $g_{\mu\nu}$). Given any two four-vectors, a^μ and b^μ, the quantity

$$a^\mu b_\mu = a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \tag{3.20}$$

is invariant (the same number in any inertial system). We shall refer to it as the scalar product of a and b; it is the four-dimensional analog to the dot product of two three-vectors (there is no four-vector analog to the cross product).* If you get tired of writing indices, feel free to use the dot notation:

$$a \cdot b \equiv a_\mu b^\mu \tag{3.21}$$

However, you will then need a way to distinguish this four-dimensional scalar product from the ordinary dot product of two three-vectors. The best way is to be scrupulously careful to put an arrow over all three-vectors (except perhaps the velocity, **v**, which, since it is not part of a four-vector, is not subject to ambiguity). In this book, I use boldface for three-vectors. Thus

$$a \cdot b = a^0 b^0 - \mathbf{a} \cdot \mathbf{b} \tag{3.22}$$

We also use the notation a^2 for the scalar product of a^μ with itself:

$$a^2 \equiv a \cdot a = (a^0)^2 - \mathbf{a}^2 \tag{3.23}$$

Notice, however, that a^2 need not be positive. Indeed, we can classify all four-vectors according to the sign of a^2:

$$\begin{aligned} &\text{If } a^2 > 0, \quad a^\mu \text{ is called } \textit{timelike} \\ &\text{If } a^2 < 0, \quad a^\mu \text{ is called spacelike} \\ &\text{If } a^2 = 0, \quad a^\mu \text{ is called } \textit{lightlike} \end{aligned} \tag{3.24}$$

From ***vectors*** it is a short step to tensors: a second-rank tensor, $s^{\mu\nu}$, carries two indices, has $4^2 = 16$ components, and transforms with two factors of **Λ**:

$$s^{\mu\nu\prime} = \Lambda^\mu_\kappa \Lambda^\nu_\sigma s^{\kappa\sigma} \tag{3.25}$$

a third-rank tensor, $t^{\mu\nu\lambda}$, has three indices, $4^3 = 64$ components, and transforms with three factors of **Λ**:

$$t^{\mu\nu\lambda\prime} = \Lambda^\mu_\kappa \Lambda^\nu_\sigma \Lambda^\lambda_\tau t^{\kappa\sigma\tau} \tag{3.26}$$

* The closest thing is $(a^\mu b^\nu - a^\nu b^\mu)$, but this is a second-rank ***tensor***, not **a** four-vector (see below).

and so on. In this hierarchy a vector is a tensor of rank 1, and a scalar (invariant) is a tensor of rank zero. We construct covariant and "mixed" tensors by lowering indices (at cost of a minus sign for each spatial index), for example

$$s^{\mu}{}_{\nu} = g_{\nu\lambda}s^{\mu\lambda}; \qquad s_{\mu\nu} = g_{\mu\kappa}g_{\nu\lambda}s^{\kappa\lambda} \tag{3.27}$$

and so on. Notice that the product of two tensors is itself a tensor [$(a^{\mu}b^{\nu})$ is a tensor of second rank; $(a^{\mu}t^{\nu\lambda\sigma})$ is a tensor of fourth rank; and so on.] Finally, we can obtain from any tensor of rank $n + 2$ a "contracted" tensor of rank n, by summing like upper and lower indices. Thus $s^{\mu}{}_{\mu}$ is a scalar; $t^{\mu\nu}{}_{\nu}$ is a vector; $a_{\mu}t^{\mu\nu\lambda}$ is a second-rank tensor.

3.3 ENERGY AND MOMENTUM

Suppose you're driving down the highway, and pretend for the sake of argument that you're going at close to the speed of light. You might want to keep an eye on two different "times": if you're worried about making an appointment in San Francisco, you should check the stationary clocks posted now and then along the side of the road. But if you're wondering when would be an appropriate time to stop for a bite to eat, it would be more sensible to look at the watch on your wrist. For according to relativity, the moving clock (in this case, your watch) is running slow (relative to the "stationary" clocks on the ground), and so too is your heart rate, your metabolism, your speech and thought, everything. Specifically, while the "ground" time advances by an infinitesimal amount dt, your own (or proper) time advances by the smaller amount $d\tau$:

$$d\tau = \frac{dt}{\gamma} \tag{3.28}$$

At normal driving speeds, of course, γ is so close to 1 that dt and $d\tau$ are essentially identical, but in elementary particle physics the distinction between laboratory time (read off the clock on the wall) and particle time (as it would appear on the particle's watch) is crucial. Although we can always get from one to the other, using equation **(3.28)**, in practice it is usually most convenient to work with proper time, because τ is invariant. All observers can read the particle's watch, and at any given moment they must all agree on what it says, even though their own clocks may differ from it and from one another.

When we speak of the "velocity" of a particle (with respect to the laboratory), we mean, of course, the distance it travels (measured in the lab frame) divided by the time it takes (measured on the lab clock):

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \tag{3.29}$$

But in view of what has just been said, it is also useful to introduce the "proper" velocity, ***q*,** which is the distance traveled (again, measured in the lab frame) divided by the proper time:*

$$\boldsymbol{\eta} \equiv \frac{d\mathbf{x}}{d\tau} \tag{3.30}$$

According to equation **(3.28),** the two velocities are related by a factor of y:

$$\boldsymbol{\eta} = \gamma \mathbf{v} \tag{3.31}$$

However, $\boldsymbol{\eta}$ is much easier to work with, for if we want to go from the lab system, *S,* to a moving system, S', both the numerator and the denominator in ***(3.29)*** must be transformed [leading to the cumbersome velocity addition rule (3.5)], whereas in equation **(3.30)** only the numerator transforms; $d\tau$, as we have seen, is invariant. In fact, proper velocity is part of a four-vector:

$$\eta^\mu = \frac{dx^\mu}{d\tau} \tag{3.32}$$

whose zeroth component is

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{(1/\gamma)dt} = \gamma c \tag{3.33}$$

Thus

$$\eta^\mu = \gamma(c, v_x, v_y, v_z) \tag{3.34}$$

Incidentally, $\eta_\mu \eta^\mu$ should be invariant, and it is:

$$\eta_\mu \eta^\mu = \gamma^2(c^2 - v_x^2 - v_y^2 - v_z^2) = \gamma^2 c^2(1 - v^2/c^2) = c^2 \tag{3.35}$$

They don't make 'em more invariant than that!

Classically, momentum is mass times velocity. We would like to carry this over in relativity, but the question arises: Which velocity should we use—ordinary velocity or proper velocity? Classical considerations offer no clue, for the two are equal in the nonrelativistic limit. In a sense, it's just a matter of definition, but there is a subtle and compelling reason why ordinary velocity would be a bad choice, whereas proper velocity is a good choice. The point is this: If we defined momentum as mv, then the law of conservation of momentum would be inconsistent with the principle of relativity (if it held in one inertial system, it would not hold in other inertial systems). But if we define momentum as mq, then conservation of momentum is consistent with the principle of relativity (if it holds in one inertial system, it automatically holds in all inertial systems). I'll let you prove this for yourself in Problem 3.10. Mind you, this doesn't guarantee

* Proper velocity is a hybrid quantity, in the sense that distance is measured in the lab frame, whereas time is measured in the particle frame. Some people object to the adjective "proper" in this context, holding that this should be reserved for quantities measured entirely in the particle frame. Of course, in its ***own*** frame the particle never moves at all—its velocity **is** zero. If my terminology disturbs you, call η the "four-velocity." I should add that although proper velocity is the more convenient quantity to calculate with, ordinary velocity is still the more natural quantity from the point of view of an observer watching a particle fly past.

that momentum *is* conserved; that's a matter for experiments to decide. But it does say that if we're hoping to extend momentum conservation to the relativistic domain, we had better not define momentum as $m\mathbf{v}$, whereas $m\eta$ is perfectly acceptable.

That's a tricky argument, and if you didn't follow it, try reading that last paragraph again. The upshot is that in relativity, momentum is defined as mass times proper velocity:

$$\mathbf{p} \equiv m\eta \tag{3.36}$$

Since proper velocity is part of a four-vector, the same goes for momentum:

$$p^\mu = m\eta^\mu \tag{3.37}$$

The spatial components of p^μ constitute the (relativistic)momentum three-vector:

$$\mathbf{p} = \text{ymv} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{3.38}$$

Meanwhile, the "time" component is

$$p^0 = \text{ymc} \tag{3.39}$$

For reasons that will appear in a moment, we define the "relativistic energy," E, as

$$\text{E} = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{3.40}$$

The zeroth component of p^μ, then, is E/c. Thus energy and momentum together make up a four-vector—the energy-momentum four-vector:

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) \tag{3.41}$$

Incidentally, from equations **(3.35)** and **(3.37)** we have

$$p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = m^2c^2 \tag{3.42}$$

which, again, is manifestly invariant.

The relativistic momentum **(3.38)** reduces to the classical expression in the nonrelativistic regime ($v \ll c$), but the same cannot be said for relativistic energy **(3.40).** To see how this quantity comes to be called "energy," we expand the radical in a Taylor series:

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right) = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots \tag{3.43}$$

Notice that the second term here corresponds to the classical kinetic energy, while the leading term (mc^2) is a constant. Now you may recall that in classical mechanics only changes in energy are physically significant—you can add a constant with impunity. In this sense the relativistic formula is consistent with the classical one, in the limit $v \ll c$ where the higher terms in the expansion are negligible. The constant term, which survives even when $v = 0$, is called the rest energy;

$$R \equiv mc^2 \tag{3.44}$$

the remainder, which is energy attributable to the motion of the particle, is the relativistic kinetic energy:

$$T \equiv mc^2(\gamma - 1) = \frac{1}{2} mv^2 + \frac{3}{8} m \frac{v^4}{c^2} + \cdots \tag{3.45}$$

(Notice that I have never mentioned relativistic mass in all this. It is a superfluous quantity that serves no useful function. In case you encounter it, the definition is $m_{rel} \equiv \gamma m$; it has died out because it differs from E only by a factor of c^2. Whatever can be said about m_{rel} could just as well be said about E, for instance, the "conservation of relativistic mass" is nothing but conservation of energy, with a factor of c^2 divided out.)

In classical mechanics there is no such thing as a massless particle; its momentum **(mv)** would be zero, its kinetic energy ($\frac{1}{2}mv^2$) would be zero, it could sustain no force, since F = ma—it would be a dynamical cipher. At first glance you might suppose that the same would be true in relativity, but a careful inspection of the formulas

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}, \qquad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{3.46}$$

reveals a loophole: When m = 0 the numerators are zero, but if $v = c$, the denominators also vanish, and these equations are indeterminate (0/0). So it is just possible that we could allow m = 0, provided the particle always travels at the speed of light. In this case equations **(3.46)** will not serve to define E and **p**; nevertheless, equation **(3.42)** presumably still applies, so that

$$E = |\mathbf{p}|c \tag{3.47}$$

for massless particles. Personally, I would regard this "argument" as a joke, were it not for the fact that at least two types of massless particles (the photon and the neutrinos) are known to exist in nature. They do indeed travel at the speed of light, and their energy and momentum are related by equation **(3.47). So** evidently we must take the loophole seriously. You may well ask: If equations **(3.46)** do not define **p** and E, what does determine the momentum and energy of a massless particle? Not the mass (that's zero by assumption); not the speed (that's always c). How, then, does a photon with an energy of 2 eV differ from a photon with an energy of **3** eV? Relativity offers no answer to this question, but curiously enough quantum mechanics does, in the form of Planck's formula:

$$E = h\nu \tag{3.48}$$

It is the frequency of the photon that determines its energy and momentum: The 2 eV photon is ***red,*** and the 3 eV photon is purple!

3.4 COLLISIONS

The reason for introducing energy and momentum is, of course, that these quantities are conserved in any physical process. In relativity, as in classical mechanics, the cleanest application of these conservation laws is to collisions. Imagine first a classical collision, in which object A hits object B (perhaps they are both carts on an air table), producing objects C and D. (See Fig. 3.2.) Of course, C and D might be the same as A and B; but we may as well allow that some paint (or whatever) rubs off A onto B, so that the final masses are not the same as the original ones. (We *do* assume, however, that A, B, C, and D are the only actors in the drama; if some wreckage, W, is left at the scene, then we would be talking about a more complicated process: $A + B \to C + D + W$.) By its nature, a collision is something that happens so fast that no external force, such as gravity, or friction with the track, has an appreciable influence. Classically, mass and momentum are always conserved in such a process; kinetic energy may or may not be conserved.

Classical Collisions

1. Mass is conserved, $m_A + m_B = m_C + m_D$.
2. Momentum is conserved, $\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C + \mathbf{p}_D$.
3. Kinetic energy may or may not be conserved.

In fact, we may distinguish three types of collisions: "sticky" ones, in which the kinetic energy decreases (typically, it is converted into heat); "explosive" ones, in which the kinetic energy increases (for example, suppose A has a compressed spring on its front bumper, and the catch is released in the course of the collision so that spring energy is converted into kinetic energy); and elastic ones, in which the kinetic energy is conserved.

Types of Collisions (Classical)

(a) Sticky: Kinetic energy decreases, $T_A + T_B > T_C + T_D$.
(b) Explosive: Kinetic energy increases, $T_A + T_B < T_C + T_D$.
(c) Elastic: Kinetic energy conserved, $T_A + T_B = T_C + T_D$.

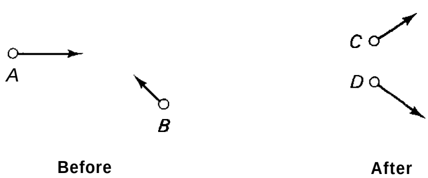

Figure 3.2 A collision in which $A + B \to C + D$.

In the extreme case of type (a), the two particles stick together, and there is really only one final object: $A + B \quad C$. In the extreme case of type (b), a single object breaks in two: $A \rightarrow C + D$ (in the language of particle physics, A decays into $C + D$).

In a relativistic collision, energy and momentum are always conserved. In other words all four components of the energy-momentum four-vector are conserved. **As** in the classical case, kinetic energy may or may not be conserved.

Relativistic Collisions

$$\left.\begin{array}{l}\textbf{1. } \text{Energy is conserved, } E_A + E_B = E_C + E_D. \\ \textbf{2. } \text{Momentum is conserved } \mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C + \mathbf{p}_D.\end{array}\right\} \Rightarrow p^\mu_A + p^\mu_B = p^\mu_C + p^\mu_D$$

3. Kinetic energy may or may not be conserved.

Again, we may classify collisions as sticky, explosive, or elastic, depending on whether the kinetic energy decreases, increases, or remains the same. Since the total energy (rest plus kinetic) is always conserved, it follows that rest energy (and hence also mass) increases in a sticky collision, decreases in an explosive collision, and is unchanged in an elastic collision.

Types of Collisions (Relativistic)

(a) Sticky: Kinetic energy decreases, rest energy and mass increase.
(b) Explosive: Kinetic energy increases, rest energy and mass decrease.
(c) Elastic: Kinetic energy, rest energy, and mass are conserved.

Please note: Except in elastic collisions, mass is not conserved;* conversely, if mass is conserved, the collision is elastic. In an explosive collision (or a particle decay), rest energy is converted into kinetic energy (or, in the absurd language of the popular press, infuriating to anyone with the slightest respect for dimensional consistency, "mass is converted into energy").

In spite of a certain structural parallel between the classical and relativistic analyses, there is a striking difference in the interpretation of inelastic collisions. In the classical case we say that energy is converted from kinetic form to some "internal" form (heat energy, spring energy, etc.), or vice versa. In the relativistic analysis we say that it goes from kinetic energy to rest energy, or vice versa. How can these possibly be consistent? After all, relativistic mechanics is supposed to reduce to classical mechanics in the limit $v \ll c$. The answer is that all "internal" forms of energy are reflected in the rest energy of an object. **A** hot potato weighs more than a cold potato; a compressed spring weighs more than a relaxed spring. On the macroscopic scale, rest energies are enormously greater than internal energies, so these mass differences are utterly negligible in everyday life, and very small even at the atomic level. Only in nuclear and particle physics are typical internal energies comparable to typical rest energies. Nevertheless, in principle, whenever you weigh an object, you are measuring not only the masses of its constituent parts, but all of their interaction energies as well.

* In the old terminology we would say that ***relativistic*** mass **is** conserved, but ***rest*** mass **is** not.

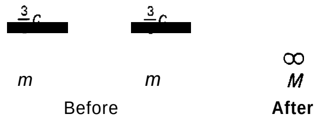

Figure 3.3 Sticky collision of two equal masses (Example 3.1).

3.5 EXAMPLES AND APPLICATIONS

Solving problems in relativistic kinematics is as much an art as a science. Although the physics involved is minimal—nothing but conservation of energy and conservation of momentum—the algebra can be formidable. Whether a given problem takes two lines or seven pages depends a lot on how skillful and experienced you are at manipulating the tools and the tricks of the trade. I now propose to work a few examples, pointing out as I go along some of the labor-saving devices that are available to you.'

EXAMPLE 3.1

Two lumps of clay, each of mass m, collide head-on at $\frac{3}{5}c$ (Fig. **3.3**). They stick together. Question: What is the mass M of the final composite lump?

Question: What is the mass, A4 of the final composite lump?

Solution. Conservation of energy says $E_1 + E_2 = E_M$. Conservation of momentum says $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_M$. In this case conservation of momentum is trivial: $\mathbf{p}_1 = -\mathbf{p}_2$, so the final lump is at rest (which was obvious from the start). The initial energies are equal, so conservation of energy yields

$$Mc^2 = 2E_m = \frac{2mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}(2mc^2)$$

Conclusion: $M = \frac{5}{2}m$. Notice that this **is** greater than the sum of the initial masses; in sticky collisions kinetic energy **is** converted into rest energy, so the mass increases.

EXAMPLE 3.2

A particle of mass M, initially at rest, decays into two pieces, each of mass m (Fig. 3.4). Question: What is the speed of each piece as it flies off?

Solution. This is, of course, the reverse of the process in Example 3.1. Conservation of momentum just says that the two lumps fly off in opposite directions at equal speeds. Conservation of energy requires that

$$M = \frac{[illegible]}{\sqrt{1-v^2/c^2}}, \quad \text{so} \quad v = c\sqrt{1-(2m/M)^2}$$

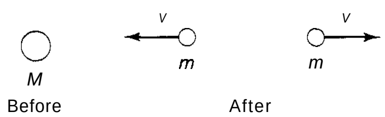

Figure 3.4 **A** particle decays into two equal pieces. (Example 3.2).

This answer makes no sense unless M exceeds 2m; there has to be at least enough rest energy available to cover the rest energies in the final state (any extra is fine; it can be soaked up in the form of kinetic energy). We say that $M = 2m$ is the threshold for the process $M \rightarrow 2m$ to occur. The deuteron, for example, is below the threshold for decay into proton plus neutron ($m_d = 1875.6$ Mev/c^2; $m_p + m_n = 1877.9$ MeV/c^2), and therefore is stable. **A** deuteron can be pulled apart, but only by pumping enough energy into the system to make up the difference. (If it puzzles you that a bound state of p and n should weigh less than the sum of its parts, the point is that the binding energy of the deuteron, which, like all internal energy, is reflected in its rest mass, is negative. Indeed, for any stable bound state the binding energy must be negative; if the composite particle weighs more than the sum of its constituents, it will spontaneously disintegrate.)

EXAMPLE 3.3

A pion at rest decays into a muon plus a neutrino (Fig. 3.5). Question: What is the speed of the muon?

Solution. Conservation of energy requires $E_\pi = E_\mu + E_\nu$. Conservation of momentum gives $\mathbf{p}_\pi = \mathbf{p}_\mu + \mathbf{p}_\nu$; but $\mathbf{p}_\pi = 0$, so $\mathbf{p}_\mu = -\mathbf{p}_\nu$. Thus the muon and the neutrino fly off back-to-back, with equal and opposite momenta.

To proceed, we need a formula relating the energy of a particle to its momentum; equation (3.42) does the job. [You might have been inclined to solve equation (3.38) for the velocity, and plug the result into equation (3.40). But that would be very poor strategy. In general, velocity is a bad parameter to work with, in relativity. Better to use equation (3.42), which takes you directly back and forth between E and **p**.]

> Suggestion *1.* To get the energy of a particle, when you know its momentum (or vice versa), use the invariant

$$E^2 - \mathbf{p}^2c^2 = m^2c^4 \tag{3.49}$$

In the present case, then:

$$E_\pi = m_\pi c^2$$
$$E_\mu = c\sqrt{m_\mu^2c^2 + \mathbf{p}_\mu^2}$$
$$E_\nu = |\mathbf{p}_\nu|c = |\mathbf{p}_\mu|c$$

Putting these into the equation for conservation of energy, we have

$$m_\pi c^2 = c\sqrt{m_\mu^2c^2 + \mathbf{p}_\mu^2} + |\mathbf{p}_\mu|c$$

or

$$(m_\pi c - |\mathbf{p}_\mu|)^2 = m_\mu^2c^2 + \mathbf{p}_\mu^2$$

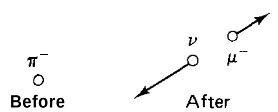

Figure 3.5 Decay of **the charged pion (Example 3.3).**

Solving for $|\mathbf{p}_\mu|$, we find

$$|\mathbf{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \mathrm{c}$$

Meanwhile, the energy of the muon [from eq. (3.49)] is

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} c^2$$

Once we know the energy and momentum of a particle, it is easy to find its velocity. If E $= \gamma m c^2$ and $\mathbf{p}$ = ***ymv,*** dividing gives

$$\mathbf{p}/E = \mathbf{v}/c^2$$

Suggestion 2. If you know the energy and momentum of a particle, and you want to determine its velocity, use

$$\mathbf{v} = \mathbf{p}c^2/E \tag{3.50}$$

So the answer to our problem is

$$v_\mu = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} c$$

Putting in the actual masses, I get $v, = 0.271c$.

There is nothing wrong with that calculation; it was a straightforward and systematic exploitation of the conservation laws. But I want to show you now a faster way to get the energy and momentum of the muon, by using four-vector notation. [I should put a superscript μ on all the four-vectors, but I don't want you to confuse the spacetime index μ with the particle identifier μ, so here, and often in the future, I will suppress the spacetime indices, and use a dot to indicate the scalar product.] Conservation of energy and momentum requires

$$p_\pi = p_\mu + p_\nu, \quad \text{or} \quad p_\nu = p_\pi - p_\mu$$

Taking the scalar product of each side with itself, we obtain

$$\mathbf{Pt} = p_\pi^2 + p_\mu^2 - 2p_\pi \cdot p_\mu$$

But

$$p_\nu^2 = 0; \qquad p_\pi^2 = m_\pi^2 c^2, \qquad p_\mu^2 = m_\mu^2 c^2; \quad \text{and} \quad p_\pi \cdot p_\mu = \frac{E_\pi}{c}\frac{E_\mu}{c} = m_\pi E_\mu$$

Therefore

$$0 = m_\pi^2 c^2 + m_\mu^2 c^2 - 2m_\pi E_\mu$$

from which E_μ follows immediately. By the same token

$$p_\mu = p_\pi - p_\nu$$

Squaring yields

$$m_\mu^2 c^2 = m_\pi^2 c^2 - 2m_\pi E_\nu$$

But $E_\nu = |\mathbf{p}_\nu|c = |\mathbf{p}_\mu|c$, so

$$2m_\pi|\mathbf{p}_\mu| = (m_\pi^2 - m_\mu^2)c$$

which gives us $|\mathbf{p}_\mu|$. In this case the problem was simple enough that the savings afforded by four-vector notation are meager, but in more complicated problems the benefits can be enormous.

> Suggestion *3.* Use four-vector notation, and exploit the invariant dot product.

One reason the use of invariants **is** so powerful in this business is that we are free to evaluate them in any inertial system we like. Frequently the laboratory frame **is** not the simplest one to work with. In a typical scattering experiment, for instance, a beam of particles is fired at a stationary target. The reaction under study might be, say, p + p → whatever, but in the laboratory the situation is asymmetrical, since one proton is moving and the other is at rest. Kinematically, the process is much simpler when viewed from a system in which the two protons approach one another with equal speeds. We call this the center-of-momentum (CM) frame, because in this system the total (three-vector) momentum is zero.

EXAMPLE 3.4

The Bevatron at Berkeley was built with the idea of producing antiprotons, by the reaction $p + p \to p + p + p + \bar{p}$. That is, a high-energy proton strikes a proton at rest, creating (in addition to the original particles) a proton-antiproton pair. Question: What is the threshold energy for this reaction (i.e., the minimum energy of the incident proton)?

Solution. In the laboratory the process looks like Figure 3.6a; in the CM frame, it looks like Figure 3.6b. Now, what **is** the condition for threshold? Answer: just barely enough incident energy to create the extra two particles. In the lab frame it is hard to see how we would formulate this condition, but in the CM it is easy: ***All*** four final particles must be at rest, with no energy "wasted" in the form of kinetic energy. (We can't have that in the

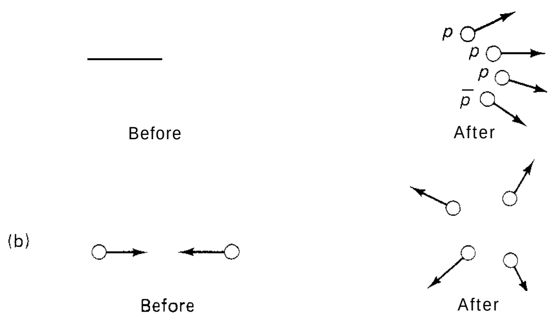

Figure 3.6 $p + p \to p + p + p + \bar{p}$. **(a)** In the lab frame; (b) in the CM frame.

lab frame, of course, since conservation of momentum requires that there be some residual motion.)

Let p^{μ}_{TOT} be the total energy-momentum four-vector in the lab; it is conserved, so it doesn't matter whether we evaluate it before or after the collision. We'll do it before:

$$p^{\mu}_{TOT} = \left(\frac{E + mc^2}{c}, |\mathbf{p}|, 0, 0\right)$$

where E and **p** are the energy and momentum of the incident proton, and m is the proton mass. Let $p^{\mu\prime}_{TOT}$ be the total energy-momentum four-vector in the CM. Again, we can evaluate it before or after the collision, this time we'll do it after:

$$p^{\mu\prime}_{TOT} = (4mc, 0, 0, 0)$$

since (at threshold) all four particles are at rest. Now $p^{\mu}_{TOT} \neq p^{\mu\prime}_{TOT}$, obviously, but the invariant products $p_{\mu TOT}p^{\mu}_{TOT}$ and $p'_{\mu TOT}p^{\mu\prime}_{TOT}$ are equal:

$$\left(\frac{E}{c} + mc\right)^2 - \mathbf{p}^2 = (4mc)^2$$

Using the standard relation (3.49) to eliminate $\mathbf{p}^2$, and solving for E, we find

$$E = 7mc^2$$

Evidently, the incident proton must carry a kinetic energy at least six times its rest energy, for this process to occur. (And in fact the first antiprotons were discovered when the machine reached about 6000 MeV.)

This is perhaps a good place to emphasize the distinction between a conserved quantity and an invariant quantity. Energy is conserved—the same value after the collision as before—but it is not invariant. Mass is invariant—the same in all inertial systems—but it is not conserved. Some quantities are both invariant and conserved; many are neither. As Example 3.4 indicates, the clever exploitation of conserved and invariant quantities can save you a lot of messy algebra. It also demonstrates that some problems are easier to analyse in the CM system, whereas others may be simpler in the lab frame.

Suggestion ***4.*** If a problem seems cumbersome in the lab frame, try analyzing it in the CM system.

Even if you're dealing with something more complicated than a collision of two identical particles, the center-of-momentum (in which $\mathbf{p}_{TOT} = 0$) is still a useful reference frame, for in this system conservation of momentum is trivial: zero before, zero after. But you might wonder whether there is always a CM frame. In other words, given a swarm of particles with masses $m_1, m_2, m_3, \ldots,$ and velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots,$ does there necessarily exist an inertial system in which the total (three-vector) momentum is zero? The answer is yes; I will prove it by finding the velocity of that frame and demonstrating that this velocity is less than c. The total energy and momentum in the lab frame (S) are

$$E_{\mathrm{TOT}} = \sum_i \gamma_i m_i c^2; \qquad \mathbf{p}_{\mathrm{TOT}} = \sum_i \gamma_i m_i \mathbf{v}_i \tag{3.51}$$

Since p^{μ}_{TOT} is a four-vector, we can use the Lorentz transformations to get the momentum in system S', moving in the direction of $\mathbf{p}_{\mathrm{TOT}}$ with speed v

$$|\mathbf{p}'_{\mathrm{TOT}}| = \gamma\left(|\mathbf{p}_{\mathrm{TOT}}| - \beta \frac{E_{\mathrm{TOT}}}{c}\right)$$

In particular, this momentum is zero if v is chosen such that

$$\frac{v}{c} = \frac{|\mathbf{p}_{\mathrm{TOT}}|c}{E_{\mathrm{TOT}}} = \frac{|\sum \gamma_i m_i \mathbf{v}_i|}{\sum \gamma_i m_i c}$$

Now, the length of the sum of three-vectors cannot exceed the sum of their lengths (this geometrically evident fact is known as the triangle inequality), so

$$\frac{v}{c} \leqslant \frac{\sum \gamma_i m_i (v_i/c)}{\sum \gamma_i m_i}$$

and since $v_i < c$, we can be sure that $v < c$.* Thus, the CM system always exists, and its velocity relative to the lab frame is given by

$$\mathbf{v}_{\mathrm{CM}} = \frac{\mathbf{p}_{\mathrm{TOT}} c^2}{E_{\mathrm{TOT}}} \tag{3.52}$$

It seems odd, looking back at the answer to Example **3.4,** that it takes an incident kinetic energy six times the proton rest energy to produce a p–$\bar{p}$ pair. After all, we're only creating $2mc^2$ of new rest energy. This example illustrates the inefficiency of scattering off a stationary target; conservation of momentum forces you to waste a lot of energy as kinetic energy in the final state. Suppose we could have fired the two protons at one another, making the laboratory itself the CM system. Then it would suffice to give each proton a kinetic energy of only mc^2, one-sixth of what the stationary-target experiment requires. This realization led, in the early 1970s, to the development of so-called colliding-beam machines (see Fig. 3.7). Today, virtually every new machine in high-energy physics is a collider.

EXAMPLE 3.5

Suppose two identical particles, each with mass m and kinetic energy T, collide head-on. Question: What is their relative kinetic energy, T' (i.e., the kinetic energy of one in the rest system of the other)?

* I am tacitly assuming that at least one of the particles is massive. If *all* of them are massless, we may obtain $v = c$, in which case there is no CM system. For example, there is no CM frame for a single photon.

Figure 3.7 Two experimental arrangements: (a) Colliding beams; (b) fixed target.

Solution. There are many ways to do this one. A quick method is to write down the total four-momentum in the CM and in the lab

$$p^{\mu}_{\text{TOT}} = \left(\frac{2E}{c}, \mathbf{0}\right), \qquad p^{\mu\prime}_{\text{TOT}} = \left(\frac{E' + mc^2}{c}, \mathbf{p}'\right)$$

set $(p_{\text{TOT}})^2 = (p'_{\text{TOT}})^2$:

$$\left(\frac{2E}{c}\right)^2 = \left(\frac{E' + mc^2}{c}\right)^2 - \mathbf{p}'^2$$

use equation **(3.49)** to eliminate $\mathbf{p}'$

$$2E^2 = mc^2(E' + mc^2)$$

and express the answer in terms of $T = E - mc^2$ and $T' = E' - mc^2$

$$T' = 4T\left(1 + \frac{T}{2mc^2}\right) \qquad \textbf{(3.53)}$$

The classical answer would have been $T' = 4T$, to which this reduces when $T \ll mc^2$. (In the rest system of *B*, *A* has, classically, twice the velocity, and hence four times as much kinetic energy as in the CM.) Now, a factor of 4 is some benefit, to be sure, but the relativistic gain can be greater by far. Colliding electrons with a laboratory kinetic energy of 1 GeV, for example, would have a relative kinetic energy of 4000 GeV!