7. Special Relativity

Although Newtonian mechanics gives an excellent description of Nature, it is not universally valid. When we reach extreme conditions — the very small, the very heavy or the very fast — the Newtonian Universe that we're used to needs replacing. You could say that Newtonian mechanics encapsulates our common sense view of the world. One of the major themes of twentieth century physics is that when you look away from our everyday world, common sense is not much use.

One such extreme is when particles travel very fast. The theory that replaces Newtonian mechanics is due to Einstein. It is called *special relativity.* The effects of special relativity become apparent only when the speeds of particles become comparable to the speed of light in the vacuum. The speed of light is

$$c = 299792458\ ms^{-1}$$

This value of c is exact. It may seem strange that the speed of light is an integer when measured in meters per second. The reason is simply that this is taken to be the definition of what we mean by a meter: it is the distance travelled by light in 1/299792458 seconds. For the purposes of this course, we'll be quite happy with the approximation $c \approx 3 \times 10^8\ ms^{-1}$.

The first thing to say is that the speed of light is fast. Really fast. The speed of sound is around 300 ms^{-1}; escape velocity from the Earth is around $10^4\ ms^{-1}$; the orbital speed of our solar system in the Milky Way galaxy is around $10^5\ ms^{-1}$. As we shall soon see, nothing travels faster than c.

The theory of special relativity rests on two experimental facts. (We will look at the evidence for these shortly). In fact, we have already met the first of these: it is simply the Galilean principle of relativity described in Section 1. The second postulate is more surprising:

- **Postulate 1:** The principle of relativity: the laws of physics are the same in all inertial frames
- **Postulate 2:** The speed of light in vacuum is the same in all inertial frames

On the face of it, the second postulate looks nonsensical. How can the speed of light look the same in all inertial frames? If light travels towards me at speed c and I run away from the light at speed v, surely I measure the speed of light as $c - v$. Right? Well, no.

This common sense view is encapsulated in the Galilean transformations that we met in Section 1.2.1. Mathematically, we derive this "obvious" result as follows: two inertial frames, S and S', which move relative to each with velocity $\mathbf{v} = (v, 0, 0)$, have Cartesian coordinates related by

$$x' = x - vt \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = t \tag{7.1}$$

If a ray of light travels in the x direction in frame S with speed c, then it traces out the trajectory $x/t = c$. The transformations above then tell us that in frame S' the trajectory if the light ray is $x'/t' = c - v$. This is the result we claimed above: the speed of light should clearly be $c - v$. If this is wrong (and it is) something must be wrong with the Galilean transformations (7.1). But what?

Our immediate goal is to find a transformation law that obeys both postulates above. As we will see, the only way to achieve this goal is to allow for a radical departure in our understanding of time. In particular, we will be forced to abandon the assumption of absolute time, enshrined in the equation $t' = t$ above. We will see that time ticks at different rates for observers sitting in different inertial frames.

7.1 Lorentz Transformations

We stick with the idea of two inertial frames, S and S', moving with relative speed v. For simplicity, we'll start by ignoring the directions y and z which are perpendicular to the direction of motion. Both inertial frames come with Cartesian coordinates: (x, t) for S and (x', t') for S'. We want to know how these are related. The most general possible relationship takes the form

$$x' = f(x, t) \quad , \quad t' = g(x, t)$$

for some function f and g. However, there are a couple of facts that we can use to immediately restrict the form of these functions. The first is that the law of inertia holds; left alone in an inertial frame, a particle will travel at constant velocity. Drawn in the (x, t) plane, the trajectory of such a particle is a straight line. Since both S and S' are inertial frames, the map $(x, t) \mapsto (x', t')$ must map straight lines to straight lines; such maps are, by definition, linear. The functions f and g must therefore be of the form

$$x' = \alpha_1 x + \alpha_2 t \quad , \quad t' = \alpha_3 x + \alpha_4 t$$

where α_i, $i = 1, 2, 3, 4$ can each be a function of v.

Secondly, we use the fact that S' is travelling at speed v relative to S. This means that an observer sitting at the origin, $x' = 0$, of S' moves along the trajectory $x = vt$ in S shown in the figure. Or, in other words, the points $x = vt$ must map to $x' = 0$. (There is actually one further assumption implicit in this statement: that the origin $x' = 0$ coincides with $x = 0$ when $t = 0$). Together with the requirement that the transformation is linear, this restricts the coefficients α_1 and α_2 above to be of the form,

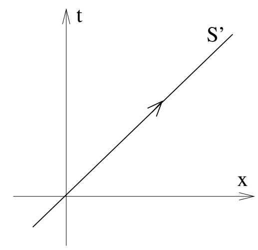

Figure 43:

$$x' = \gamma(x - vt) \tag{7.2}$$

for some coefficient γ. Once again, the overall coefficient γ can be a function of the velocity: $\gamma = \gamma_v$. (We've used subscript notation γ_v rather than the more standard $\gamma(v)$ to denote that γ depends on v. This avoids confusion with the factors of $(x - vt)$ which aren't arguments of γ but will frequently appear after γ like in the equation (7.2)).

There is actually a small, but important, restriction on the form of γ_v: it must be an even function, so that $\gamma_v = \gamma_{-v}$. There are a couple of ways to see this. The first is by using rotational invariance, which states that γ can depend only on the direction of the relative velocity $\mathbf{v}$, but only on the magnitude $v^2 = \mathbf{v} \cdot \mathbf{v}$. Alternatively, if this is a little slick, we can reach the same conclusion by considering inertial frames $\tilde{S}$ and $\tilde{S}'$ which are identical to S and S' except that we measure the x-coordinate in the opposite direction, meaning $\tilde{x} = -x$ and $\tilde{x}' = -x'$. While S is moving with velocity $+v$ relative to S', $\tilde{S}$ is moving with velocity $-v$ with respect to $\tilde{S}'$ simply because we measure things in the opposite direction. That means that

$$\tilde{x}' = \gamma_{-v}\left(\tilde{x} + v\tilde{t}\right)$$

Comparing this to (7.2), we see that we must have $\gamma_v = \gamma_{-v}$ as claimed.

We can also look at things from the perspective of S', relative to which the frame S moves backwards with velocity $-v$. The same argument that led us to (7.2) now tells us that

$$x = \gamma(x' + vt') \tag{7.3}$$

Now the function $\gamma = \gamma_{-v}$. But by the argument above, we know that $\gamma_v = \gamma_{-v}$. In other words, the coefficient γ appearing in (7.3) is the same as that appearing in (7.2).

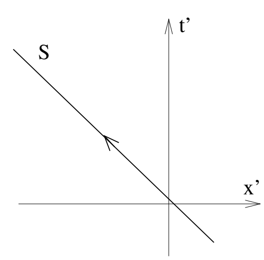

Figure 44:

At this point, things don't look too different from what we've seen before. Indeed, if we now insisted on absolute time, so $t = t'$, we're forced to have $\gamma = 1$ and we get back to the Galilean transformations (7.1). However, as we've seen, this is not compatible with the second postulate of special relativity. So let's push forward and insist instead that the speed of light is equal to c in both S and S'. In S, a light ray has trajectory

$$x = ct$$

While, in S', we demand that the same light ray has trajectory

$$x' = ct'$$

Substituting these trajectories into (7.2) and (7.3), we have two equations relating t and t',

$$ct' = \gamma(c - v)t \quad \text{and} \quad ct = \gamma(c + v)t'$$

A little algebra shows that these two equations are compatible only if γ is given by

$$\gamma = \sqrt{\frac{1}{1 - v^2/c^2}} \tag{7.4}$$

We'll be seeing a lot of this coefficient γ in what follows. Notice that for $v \ll c$, we have $\gamma \approx 1$ and the transformation law (7.2) is approximately the same as the Galilean transformation (7.1). However, as $v \to c$ we have $\gamma \to \infty$. Furthermore, γ becomes imaginary for $v > c$ which means that we're unable to make sense of inertial frames with relative speed $v > c$.

Equations (7.2) and (7.4) give us the transformation law for the spatial coordinate. But what about for time? In fact, the temporal transformation law is already lurking in our analysis above. Substituting the expression for x' in (7.2) into (7.3) and rearranging, we get

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{7.5}$$

We shall soon see that this equation has dramatic consequences. For now, however, we merely note that when $v \ll c$, we recover the trivial Galilean transformation law $t' \approx t$. Equations (7.2) and (7.5) are the *Lorentz transformations*.

7.1.1 Lorentz Transformations in Three Spatial Dimensions

In the above derivation, we ignored the transformation of the coordinates y and z perpendicular to the relative motion. In fact, these transformations are trivial. Using the above arguments for linearity and the fact that the origins coincide at $t = 0$, the most general form of the transformation is

$$y' = \kappa y$$

But, by symmetry, we must also have $y = \kappa y'$. Clearly, we require $\kappa = 1$. (The other possibility $\kappa = -1$ does not give the identity transformation when $v = 0$. Instead, it is a reflection).

With this we can write down the final form of the Lorentz transformations. Note that they look more symmetric between x and t if we write them using the combination ct,

$$\begin{aligned} x' &= \gamma\left(x - \frac{v}{c}\,ct\right) \\ y' &= y \\ z' &= z \\ ct' &= \gamma\left(ct - \frac{v}{c}x\right) \end{aligned} \tag{7.6}$$

where γ is given by (7.4). These are also known as *Lorentz boosts*. Notice that for $v/c \ll 1$, the Lorentz boosts reduce to the more intuitive Galilean boosts that we saw in Section 1. (We sometimes say, rather sloppily, that the Lorentz transformations reduce to the Galilean transformations in the limit $c \to \infty$).

It's also worth stressing again the special properties of these transformations. To be compatible with the first postulate, the transformations must take the same form if we invert them to express x and t in terms of x' and t', except with v replaced by $-v$. And, after a little bit of algebraic magic, they do.

Secondly, we want the speed of light to be the same in all inertial frames. For light travelling in the x direction, we already imposed this in our derivation of the Lorentz transformations. But it's simple to check again: in frame S, the trajectory of an object travelling at the speed of light obeys $x = ct$. In S', the same object will follow the trajectory $x' = \gamma(x - vt) = \gamma(ct - vx/c) = ct'$.

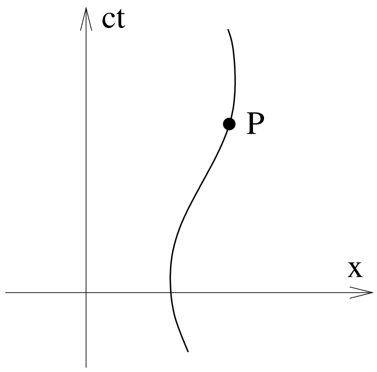

Figure 45: The worldline of a particle

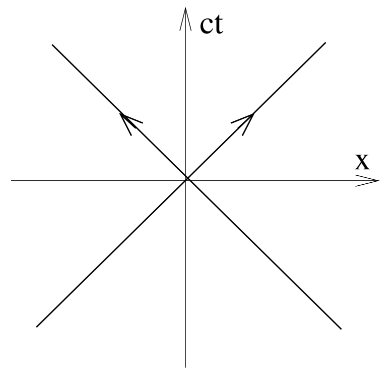

Figure 46: Light rays travel at 45°

What about an object travelling in the y direction at the speed of light? Its trajectory in S is $y = ct$. From (7.6), its trajectory in S' is $y' = ct'/\gamma$ and $x' = -vt'$. Its speed in S' is therefore $v'^2 = v_x^2 + v_y^2$, or

$$v'^2 = \left(\frac{x'}{t'}\right)^2 + \left(\frac{y'}{t'}\right)^2 = v^2 + \frac{c^2}{\gamma^2} = c^2$$

7.1.2 Spacetime Diagrams

We'll find it very useful to introduce a simple spacetime diagram to illustrate the physics of relativity. In a fixed inertial frame, S, we draw one direction of space — say x — along the horizontal axis and time on the vertical axis. But things look much nicer if we rescale time and plot ct on the vertical instead. In the context of special relativity, space and time is called *Minkowski space*. (Although the true definition of Minkowski space requires some extra structure on space and time which we will meet in Section (7.3)).

This is a spacetime diagram. Each point, P, represents an *event*. In the following, we'll label points on the spacetime diagram as coordinates (ct, x) i.e. giving the coordinate along the vertical axis first. This is backwards from the usual way coordinates but is chosen so that it is consistent with a later, standard, convention that we will meet in Section 7.3.

A particle moving in spacetime traces out a curve called a *worldline* as shown in the figure. Because we've rescaled the time axis, a light ray moving in the x direction moves at 45°. We'll later see that no object can move faster than the speed of light which means that the worldlines of particles must always move upwards at an angle steeper than 45°.

The horizontal and vertical axis in the spacetime diagram are the coordinates of the inertial frame S. But we could also draw the axes corresponding to an inertial frame S' moving with relative velocity $\mathbf{v} = (v, 0, 0)$. The t' axis sits at $x' = 0$ and is given by

$$x = vt$$

Meanwhile, the x' axis is determined by $t' = 0$ which, from the Lorentz transformation (7.6), is given by the equation

$$ct = \frac{v}{c}x$$

These two axes are drawn on the figure to the right. They can be the thought of as the x and ct axes, rotated by an equal amount towards the diagonal light ray. The fact the axes are symmetric about the light ray reflects the fact that the speed of light is equal to c in both frames.

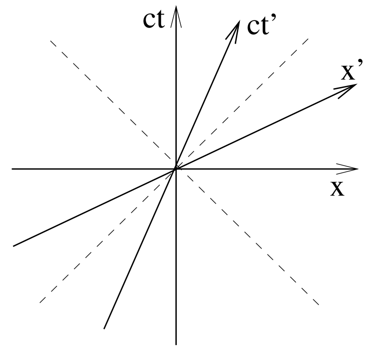

Figure 47:

7.1.3 A History of Light Speed

The first evidence that light does not travel instantaneously was presented by the Danish Astronomer Ole Rømer in 1676. He noticed that the periods of the orbits of Io, the innermost moon of Jupiter, are not constant. When the Earth is moving towards Jupiter, the orbits are a few minutes shorter; when the Earth moves away, the orbits are longer by the same amount. Rømer correctly deduced that this was due to the finite speed of light and gave a rough estimate for the value of c.

By the mid 1800s, the speed of light had been determined fairly accurately using experiments involving rotating mirrors. Then came a theoretical bombshell. Maxwell showed that light could be understood as oscillations of the electric and magnetic fields. He related the speed of light to two constants, ϵ_0 and μ_0, the permittivity and permeability of free space, that arise in the theory of electromagnetism,

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \tag{7.7}$$

But, as we have seen, Newtonian physics tells us that speeds are relative. If Maxwell's equations predict a value for the speed of light, it was thought that these equations must be valid only in a preferred reference frame. Moreover, this does not seem unreasonable; if light is a wave then surely there is something waving. Just as water waves need water, and sound waves need air, so it was thought that light waves need a material to propagate in. This material was dubbed the *luminiferous ether* and it was thought that Maxwell's equations must only be valid in the frame at rest with respect to this ether.

In 1881, Michelson and Morley performed an experiment to detect the relative motion of the Earth through the ether. Since the Earth is orbiting the Sun at a speed of $3 \times 10^4\ ms^{-1}$, even if it happens to be stationary with respect to the ether at some point, six months later this can no longer be the case.

Suppose that at some moment the Earth is moving in the x-direction relative to the ether with some speed v. The Newtonian addition of velocities tells us that light propagating in the x-direction should have speed $c+v$ going one way and $c-v$ going the other. The total time to travel backwards and forwards along a length L should therefore be

$$T_x = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2cL}{c^2 - v^2}$$

Meanwhile, light making the same journey in the y-direction will have to travel (by Pythagoras) a total distance of $\sqrt{L^2 + v^2(T_y/2)^2}$ on each leg of the journey. It makes this journey at speed c, meaning that we can equate

$$\frac{cT_y}{2} = \sqrt{L^2 + v^2(T_y/2)^2} \qquad \Rightarrow \qquad T_y = \frac{2L}{\sqrt{c^2 - v^2}}$$

The goal of the Michelson-Morley experiment was to measure the time difference between T_y and T_x using interference patterns of light ray making the two journeys. Needless to say, the experiment didn't work: there seemed to be no difference in the time taken to travel in the x direction and y direction.

Towards the end of the 1800s, the null result of the Michelson-Morley experiment had become one of the major problems in theoretical physics. Several explanations were proposed, including the idea that the ether was somehow dragged along with the Earth. The Dutch physicist, Hendrik Lorentz, went some way to finding the correct solution. He had noticed that Maxwell's equations had the peculiar symmetry that we now call the Lorentz transformations. He argued that if a reason could be found that would allow distances between matter to change as

$$x' = \gamma(x - vt)$$

then lengths would be squeezed in the direction parallel to the ether, explaining why no difference is seen between T_x and T_y. (We will shortly derive this contraction of lengths using special relativity). Lorentz set to work trying to provide a mechanical explanation for this transformation law.

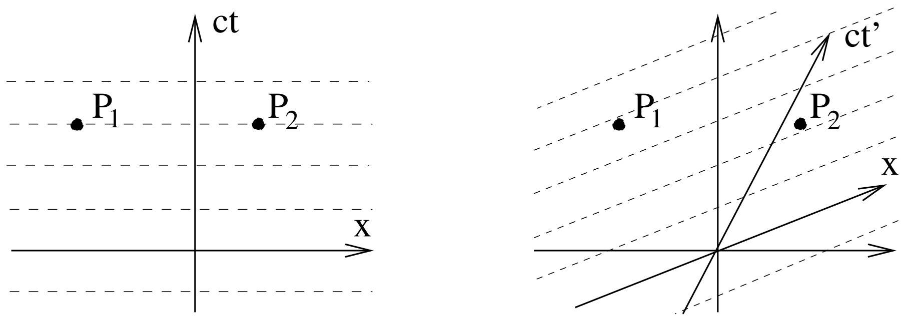

Figure 48: Simultaneity is relative

Although Lorentz had put in place much of the mathematics, the real insight came from Einstein in 1905. He understood that there is no mechanical mechanism underlying the Lorentz transformations. Nor is there an ether. Instead, the Lorentz transformations are a property of space and time themselves.

With Einstein's new take on the principle of relativity, all problems with Maxwell's equation evaporate. There is no preferred inertial frame. Instead, Maxwell's equations work equally well in all inertial frames. However, they are not invariant under the older transformations of Galilean relativity; instead they are the first law of physics to be invariant under the correct transformations (7.6) of Einstein/Lorentz relativity. It's worth pointing out that, from this perspective, we could dispense with the second postulate of relativity all together. We need only insist that the laws of physics – which include Maxwell's equations – hold in all inertial frames. Since Maxwell's equations predict (7.7), this implies the statement that the speed of light is the same in all inertial frames. But since we haven't yet seen the relationship between Maxwell's equations, light and relativity, it's perhaps best to retain the second postulate for now.

7.2 Relativistic Physics

In this section we will explore some of the more interesting and surprising consequences of the Lorentz transformations.

7.2.1 Simultaneity

We start with a simple question: how can we be sure that things happen at the same time? In Newtonian physics, this is a simple question to answer. In that case, we have an absolute time t and two events, P_1 and P_2, happen at the same time if $t_1 = t_2$. However, in the relativistic world, things are not so easy.

We start with an observer in inertial frame S, with time coordinate t. This observer sensibly decides that two events, P_1 and P_2, occur simultaneously if $t_1 = t_2$. In the spacetime diagram on the left of Figure 48 we have drawn lines of simultaneity for this observer.

But for an observer in the inertial frame S', simultaneity of events occurs for equal t'. Using the Lorentz transformation, lines of constant t' become lines described by the equation $t - vx/c^2 =$ constant. These lines are drawn on the spacetime diagram on the right of Figure 48.

The upshot of this is that two events simultaneous in one inertial frame are not simultaneous in another. An observer in S thinks that events P_1 and P_2 happen at the same time. All other observers disagree.

A Train Story

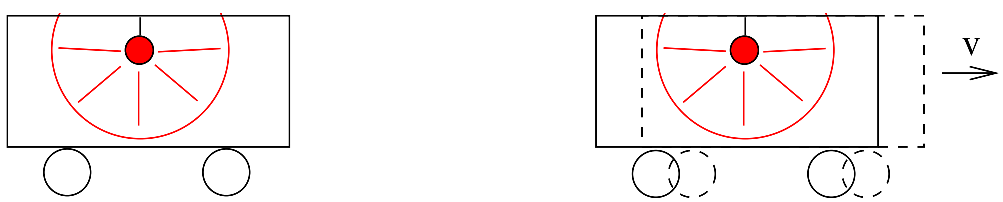

Figure 49: Lights on Trains: Simultaneity is Relative

The fact that all observers cannot agree on what events are simultaneous is a direct consequence of the fact that all observers do agree on the speed of light. We can illustrate this connection with a simple *gedankenexperiment*. (An ugly German word for "thought experiment", a favourite trick of theoretical physicists who can't be bothered to do real experiments). Consider a train moving at constant speed, with a lightbulb hanging from the middle of one of the carriages. A passenger on the train turns on the bulb and, because the bulb is equidistant from both the front and back wall of the carriage, observes that the light hits both walls at the same time.

However, a person standing on the platform as the train passes through disagrees. The light from the bulb travels at equal speed $\pm c$ to the left and right, but the back of the train is rushing towards the point in space where the light first emerged from. The person on the platform will see the light hit the back of the train first.

It is worth mentioning that although the two people disagree on whether the light hits the walls at the same time, this does not mean that they can't be friends.

A Potential Confusion: What the Observer Observes

We'll pause briefly to press home a point that may lead to confusion. You might think that the question of simultaneity has something to do with the finite speed of propagation. You don't see something until the light has travelled to you, just as you don't hear something until the sound has travelled to you. This is *not* what's going on here! A look at the spacetime diagram in Figure 48 shows that we've already taken this into account when deciding whether two events occur simultaneously. The lack of simultaneity between moving observers is a much deeper issue, not due to the finiteness of the speed of light but rather due to the constancy of the speed of light.

The confusion about the time of flight of the signal is sometimes compounded by the common use of the word *observer* to mean "inertial frame". This brings to mind some guy sitting at the origin, surveying all around him. Instead, you should think of the observer more as a Big Brother figure: a sea of clocks and rulers throughout the inertial frame which can faithfully record and store the position and time of any event, to be studied at some time in the future.

Of course, this means that there is a second question we can ask which is: what does the guy sitting at the origin actually see? Now we have to take into account both the relative nature of simultaneity *and* the issues related with the finite speed of propagation. This adds an extra layer of complexity which we will discuss in Section 7.6.

7.2.2 Causality

We've seen that different observers disagree on the temporal ordering of two events. But where does that leave the idea of causality? Surely it's important that we can say that one event definitely occurred before another. Thankfully, all is not lost: there are only some events which observers can disagree about.

To see this, note that because Lorentz boosts are only possible for $v < c$, the lines of simultaneity cannot be steeper than $45°$. Take a point P and draw the $45°$ light rays that emerge from P. This is called the *light cone*. (For once, in the figure, I've drawn this with an extra spatial dimension present to illustrate how this works in spatial dimensions bigger than one). The light cone is really two cones, touching at the point P. They are known as the future light cone and past light cone.

For events inside the light cone of P, there is no difficulty deciding on the temporal ordering of events. All observers will agree that Q occurred after P. However, for events outside the light cone, the matter is up for grabs: some observers will see R as happening after P; some before.

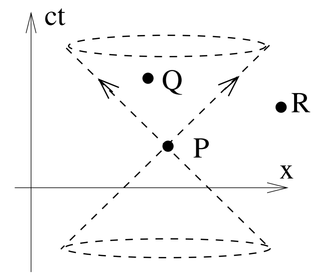

Figure 50:

This tells us that the events which all observers agree can be causally influenced by P are those inside the future light cone. Similarly, the events which can plausibly influence P are those inside the past light cone. This means that we can sleep comfortably at night, happy in the knowledge that causality is preserved, only if nothing can propagate outside the light cone. But that's the same thing as travelling faster than the speed of light.

The converse to this is that if we do ever see particles that travel faster than the speed of light, we're in trouble. We could use them to transmit information faster than light. But another observer would view this as transmitting information backwards in time. All our ideas of cause and effect will be turned on their head. You will therefore be relieved to learn that we will show in Section 7.3 why it is impossible to accelerate particles past the light speed barrier.

There is a corollary to the statement that events outside the lightcone cannot influence each other: there are no perfectly rigid objects. Suppose that you push on one end of a rod. The other end cannot move immediately since that would allow us to communicate faster than the speed of light. Of course, for real rods, the other end does not move instantaneously. Instead, pushing on one end of the rod initiates a sound wave which propagates through the rod, telling the other parts to move. The statement that there is no rigid object is simply the statement that this sound wave must travel slower than the speed of light.

Finally, let me mention that when we're talking about waves, as opposed to point particles, there is a slight subtlety in exactly what must travel slower than light. There are at least two velocities associated to a wave: the group velocity is (usually) the speed at which information can be communicated. This is less than c. In contrast, the phase velocity is the speed at which the peaks of the wave travel. This can be greater than c, but transmits no information.

7.2.3 Time Dilation

We'll now turn to one of the more dramatic results of special relativity. Consider a clock sitting stationary in the frame S' which ticks at intervals of T'. This means that the tick events in frame S' occur at $(ct'_1, 0)$ then $(ct'_1 + cT', 0)$ and so on. What are the intervals between ticks in frame S?

We can answer immediately from the Lorentz transformations (7.6). Inverting this gives

$$t = \gamma \left(t' + \frac{vx'}{c^2} \right)$$

The clock sits at $x' = 0$, so we immediately learn that in frame S, the interval between ticks is

$$T = \gamma T'$$

This means that the gap between ticks is longer in the stationary frame. A moving clock runs more slowly. But the same argument holds for any process, be it clocks, elementary particles or human hearts. The correct interpretation is that time itself runs more slowly in moving frames.

Another Train Story

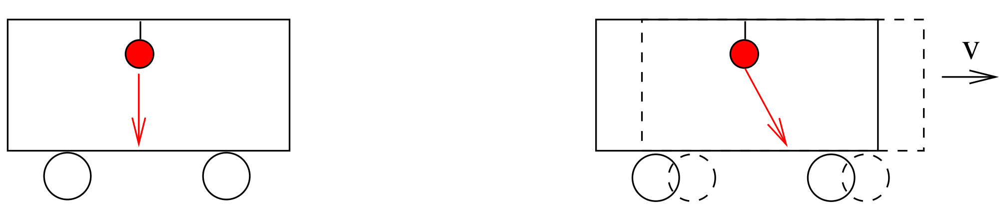

Figure 51: More Lights on Trains: Time Dilation

Let's go back to our lightbulb and gedankenbahn. If the train has height h, a passenger on the train will measure time $t' = h/c$ for the light to travel from the light bulb to the middle of the floor (i.e. the point directly below the light bulb). What about for the guy on the platform? After the light turns on, the train has moved forward at speed v. To hit the same point on the floor, the light has to travel a distance $\sqrt{h^2 + (vt)^2}$. The time taken is therefore

$$t = \frac{\sqrt{h^2 + (vt)^2}}{c} \quad \Rightarrow \quad t = \frac{h}{c}\sqrt{\frac{1}{1 - v^2/c^2}} = \gamma t'$$

This gives another, more pictorial, derivation of the time dilation formula.

On Muons and Planes

Away from the world of gedankenexperiments, there are a couple of real experimental consequences of time dilation. Certainly the place that this phenomenon is tested most accurately is in particle accelerators where elementary particles routinely reach speeds close to c. The protons spinning around the LHC have $\gamma \approx 3500$. The previous collider in CERN, called LEP, accelerated electrons and positrons to $\gamma \approx 2 \times 10^5$. (Although the electrons in LEP were travelling faster than the protons in LHC, the greater mass of the protons means that there is substantially more energy in the LHC collisions).

The effect of time dilation is particularly vivid on unstable particles which live much longer in the lab frame than in their own rest frame. An early demonstration was seen in *muons* in 1941. These are heavier, unstable, versions of the electron. They decay into an electron, together with a couple of neutrinos, with a half-life of $\tau \approx 2 \times 10^{-6}$ s. Muons are created when cosmic rays hit the atmosphere, and subsequently rain down on Earth. Yet to make it down to sea level, it takes about $t = 7 \times 10^{-6}$ s, somewhat longer than their lifetime. Given this, why are there any muons detected on Earth at all? Surely they should have decayed. The reason that they do not is because the muons are travelling at a speed $v \approx 0.99c$, giving $\gamma \approx 10$. From the muon's perspective, the journey only takes $t' = t/\gamma \approx 7 \times 10^{-7}$ s, somewhat less than their lifetime.

Note that elementary particles are, by definition, structureless. They're certainly not some clock with an internal machinery. The reason that they live longer can't be explained because of some mechanical device which slows down: it is time itself which is running slower.

A more direct test of time dilation was performed in 1971 by Hafele and Keating. They flew two atomic clocks around the world on commercial airliners; two more were left at home. When they were subsequently brought together, their times differed by about 10^{-7} s. There are actually two contributions to this effect: the time dilation of special relativity that we've seen above, together with a related effect in general relativity due to the gravity of the Earth.

Twin Paradox

Two twins, Luke and Leia, decide to spend some time apart. Leia stays at home while Luke jumps in a spaceship and heads at some speed v to the planet Tatooine. With sadness, Leia watches Luke leave but is relieved to see — only a time T later from her perspective — him safely reach the planet.

However, upon arrival, Luke finds that he doesn't like Tatooine so much. It is a dusty, violent place with little to do. So he turns around and heads back to Leia at the same speed v as before. When he returns, he finds that Leia has aged by $T_{\rm Leia} = 2T$. And yet, fresh faced Luke has only aged by $T_{\rm Luke} = 2T/\gamma$. We see, that after the journey, Luke is younger than Leia. In fact, for large enough values of γ, Luke could return to find Leia long dead.

This is nothing more than the usual time dilation story. So why is it a paradox? Well, things seem puzzling from Luke's perspective. He's sitting happily in his inertial spaceship, watching Leia and the whole planet flying off into space at speed v. From his perspective, it should be Leia who is younger. Surely things should be symmetric between the two?

The resolution to this "paradox" is that there is no symmetry between Luke's journey and Leia's. Leia remained in an inertial frame for all time. Luke, however, does not. When he reaches Tatooine, he has to turn around and this event means that he has to accelerate. This is what breaks the symmetry.

We can look at this in some detail. We draw the spacetime diagram in Leia's frame. Luke sits at $x = vt$, or $x' = 0$. Leia sits at $x = 0$. Luke reaches Tatooine at point P. We've also drawn two lines of simultaneity. The point Y is when Leia thinks that Luke has arrived on Tatooine. The point X is where Luke thinks Leia was when he arrived at Tatooine. As we've already seen, it's quite ok for Luke and Leia to disagree on the simultaneity of these points. Let's figure out the coordinates for X and Y.

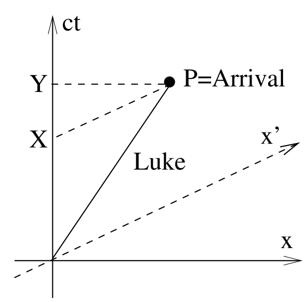

Figure 52:

Event Y sits at coordinate $(cT, 0)$ in Leia's frame, while P is at (cT, vT). The time elapsed in Luke's frame is just the usual time dilation calculation,

$$T' = \gamma\left(T - \frac{v^2 T}{c^2}\right) = \frac{T}{\gamma}$$

We can also work out the coordinates of the event X. Clearly this takes place at $x = 0$ in Leia's frame. In Luke's frame, this is simultaneous with his arrival at Tatooine, so occurs at $t' = T' = T/\gamma$. We can again use the Lorentz transformation

$$t' = \gamma\left(t - \frac{v^2 x}{c^2}\right)$$

now viewed as an equation for t given x and t'. This gives us

$$t = \frac{T'}{\gamma} = \frac{T}{\gamma^2}$$

So at this point, we see that everything is indeed symmetric. When Luke reaches Tatooine, he thinks that Leia is younger than him by a factor of γ. Meanwhile, Leia thinks that Luke is younger than her by the same factor .

Things change when Luke turns around. To illustrate this, let's first consider a different scenario where he doesn't return from Tatooine. Instead, as soon as he arrives, he synchronises his clock with a friend – let's call him Han – who is on his way to meet Leia. Now things are still symmetric. Luke thinks that Leia has aged by T/γ^2 on the outward journey; Han also thinks that Leia has aged by T/γ^2 on the inward journey. So where did the missing time go?

We can see this by looking at the spacetime diagram of Han's journey. We've again drawn lines of simultaneity. From Han's perspective, he thinks that Leia is sitting at point Z when he leaves Tatooine, while Luke is still convinced that she's sitting at point X. It's not hard to check that at point Z, Leia's clock reads $t = 2T - T/\gamma^2$.

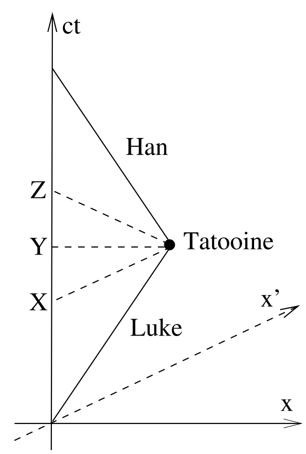

Figure 53:

From this perspective, we can also see what happens if Luke does return home. When he arrives at Tatooine, he thinks Leia is at point X. Yet, in the time he takes to turn around and head home, the acceleration makes her appear to rapidly age, from point X to point Z.

7.2.4 Length Contraction

We've seen that moving clocks run slow. We will now show that moving rods are shortened. Consider a rod of length L' sitting stationary in the frame S'. What is its length in frame S?

To begin, we should state more carefully something which seems obvious: when we say that a rod has length L', it means that the distance between the two end points *at equal times* is L'. So, drawing the axes for the frame S', the situation looks like the picture on the left. The two, simultaneous, end points in S' are P_1 and P_2. Their coordinates in S' are $(ct', x') = (0, 0)$ and $(0, L')$ respectively.

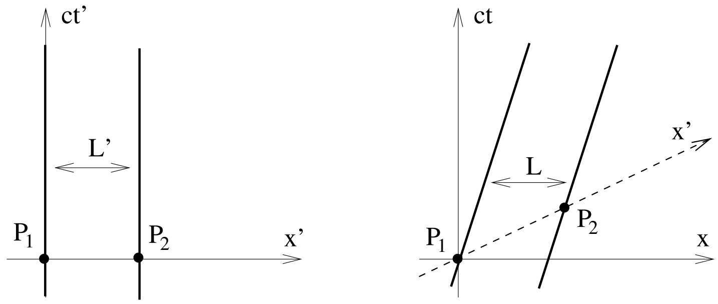

Figure 54: Length Contraction

Now let's look at this in frame S. This is drawn in right-hand picture. Clearly P_1 sits at $(ct, x) = (0, 0)$. Meanwhile, the Lorentz transformation gives us the coordinate for P_2

$$x = \gamma L' \quad \text{and} \quad t = \frac{\gamma v L'}{c^2}$$

But to measure the rod in frame S, we want both ends to be at the same time. And the points P_1 and P_2 are not simultaneous in S. We can follow the point P_2 backwards along the trajectory of the end point to Q_2, which sits at

$$x = \gamma L' - vt$$

We want Q_2 to be simultaneous with P_1 in frame S. This means we must move back a time $t = \gamma v L/c^2$, giving

$$x = \gamma L' - \frac{\gamma v^2 L'}{c^2} = \frac{L'}{\gamma}$$

This is telling us that the length L measured in frame S is

$$L = \frac{L'}{\gamma}$$

It is shorter than the length of the rod in its rest frame by a factor of γ. This phenomenon is known as *Lorentz contraction.*

Putting Ladders in Barns

Take a ladder of length $2L$ and try to put it in a barn of length L. If you run fast enough, can you squeeze it? Here are two arguments, each giving the opposite conclusion

- From the perspective of the barn, the ladder contracts to a length $2L/\gamma$. This shows that it can happily fit inside as long as you run fast enough, with $\gamma \geq 2$
- From the perspective of the ladder, the barn has contracted to length L/γ. This means there's no way you're going to get the ladder inside the barn. Running faster will only make things worse

What's going on? As usual, to reconcile these two points of view we need to think more carefully about the question we're asking. What does it mean to "fit a ladder inside a barn"? Any observer will agree that we've achieved this if the back end gets in the door before the front end hits the far wall. But we know that simultaneity of events is not fixed, so the word "before" in this definition suggests that it may be something different observers will disagree on. Let's see how this works.

The spacetime diagram in the frame of the barn is drawn in the figure with $\gamma > 2$. We see that, from the barn's perspective, both back and front ends of the ladder are happily inside the barn at the same time. We've also drawn the line of simultaneity for the ladder's frame. This shows that when the front of the ladder hits the far wall, the back end of the ladder has not yet got in the door. Is the ladder in the barn? Well, it all depends who you ask.

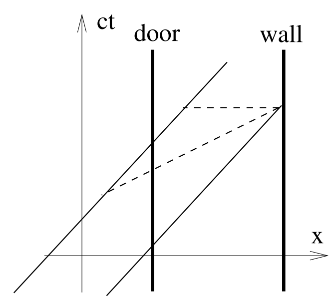

Figure 55: