V. Important Topic.

**

Euler's equation

(1) Order 2 : $ax^2y'' + bxy' + cy = 0$.

Procedure : Put $v = \ln x$

Then $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$ (Chain Rule)

$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dv}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dv}$ — (1).

Differentiating again w.r.t x,

$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dv^2} \cdot \frac{dv}{dx}$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dv^2}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dv^2} - \frac{dy}{dv}$ — (2) (using (1))

$\therefore$ $x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$ } where $D \equiv \frac{d}{dv}$

(2) Order 3 : $ax^3y''' + bx^2y'' + cxy' + ey = 0$

Proceeding as in Order 2,

$xy' = Dy$

$x^2y'' = D(D-1)y$

& $x^3y''' = D(D-1)(D-2)y$

where $D \equiv \frac{d}{dv}$.

classmate
Date
Page

Examples

① $x^2y'' + xy' + 9y = 0$.

Put $v = \ln x$

Then $\left.\begin{aligned} xy' &= Dy \\ x^2y'' &= D(D-1)y \end{aligned}\right\}$ where $D = \frac{d}{dv}$.

The eq^n becomes

$D(D-1)y + Dy + 9y = 0$

$\Rightarrow (D^2 - D + D + 9)y = 0$

$\Rightarrow (D^2 + 9)y = 0$

$\Rightarrow \frac{d^2y}{dv^2} + 9y = 0$. (L.D.E with C.C

A.E. $r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$\Rightarrow \quad y = c_1 \cos 3v + c_2 \sin 3v$

$\Rightarrow \quad y = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)$

is the solution.

(2) $\quad x^3 y''' - 3x^2 y'' + xy' = 0.$

Put $v = \ln x$. Then the eqn becomes

$$\left(D(D-1)(D-2) - 3D(D-1) + D\right) y = 0$$

$$\Rightarrow \left((D^2 - D)(D-2) - 3D^2 + 3D + D\right) y = 0$$

$$\Rightarrow \left(D^3 - 2D^2 - D^2 + 2D - 3D^2 + 3D + D\right) y = 0$$

$$\Rightarrow \left(D^3 - 6D^2 + 6D\right) y = 0.$$

A.E: $\quad r^3 - 6r^2 + 6r = 0$

$\Rightarrow \quad r = 0, \quad r^2 - 6r + 6 = 0$

$\Rightarrow \quad r = 0, \quad r = 3 \pm \sqrt{3}$

so $\quad y = c_1 + c_2 e^{(3+\sqrt{3})v} + c_3 e^{(3-\sqrt{3})v}$

$\Rightarrow \quad y = c_1 + c_2 x^{(3+\sqrt{3})} + c_3 x^{(3-\sqrt{3})}$

$\Rightarrow \quad y = c_1 + x^3 \left(c_2 x^{\sqrt{3}} + c_3 x^{-\sqrt{3}}\right)$ is the required solutio

Assignment :

Ex 2.1 → Q52 – 55

Ex 2.3 → Q52 – 58

at if $x > 0$,
Eq. (22) into

(23)

$_1$ and r_2 of
nd distinct,

Make the substitution $v = \ln x$ of Problem 51 to find general solutions (for $x > 0$) of the Euler equations in Problems 52–56.

52. $x^2y'' + xy' - y = 0$

53. $x^2y'' + 2xy' - 12y = 0$

54. $4x^2y'' + 8xy' - 3y = 0$

55. $x^2y'' + xy' = 0$

56. $x^2y'' - 3xy' + 4y = 0$

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$^{(3)}(0) = 30.$

(25)

inearly indepen-
all x such that
$x \neq 0$;
$x = 0$;
$0) = 0$.
by one formula
graphs of these

$y_2(x)$ in Problem 50.

51. According to Problem 51 in Section 2.1, the substitution $v = \ln x$ $(x > 0)$ transforms the second-order Euler equation $ax^2y'' + bxy' + cy = 0$ to a constant-coefficient homogeneous linear equation. Show similarly that this same substitution transforms the third-order Euler equation

$$ax^3y''' + bx^2y'' + cxy' + dy = 0$$

(where a, b, c, d are constants) into the constant-coefficient equation

$$a\frac{d^3y}{dv^3} + (b - 3a)\frac{d^2y}{dv^2} + (c - b + 2a)\frac{dy}{dv} + dy = 0.$$

Make the substitution $v = \ln x$ of Problem 51 to find general solutions (for $x > 0$) of the Euler equations in Problems 52 through 58.

52. $x^2y'' + xy' + 9y = 0$

53. $x^2y'' + 7xy' + 25y = 0$

54. $x^3y''' + 6x^2y'' + 4xy' = 0$

55. $x^3y''' - x^2y'' + xy' = 0$

56. $x^3y''' + 3x^2y'' + xy' = 0$

57. $x^3y''' - 3x^2y'' + xy' = 0$

58. $x^3y''' + 6x^2y'' + 7xy' + y = 0$