HYPERBOLA : A hyperbola is the set of all points in the plane, the difference of whose distances from two fixed points is a given positive constant that is less than the distance between the fixed points.

The two fixed points are called the $\boxed{\text{foci}}$ of the hyperbola.

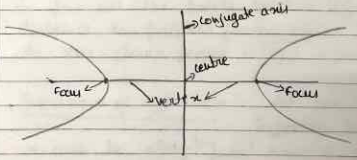
Note: ① It is understood that the term difference used in defn means the distance to the farther focus minus the distance to the closer focus.

② Thus, the points on the hyperbola form two branches each wrapping around the closer focus.

The mid point of the line segment joining the foci is called the $\boxed{\text{centre}}$ of the hyperbola.

The line through the foci is called the $\boxed{\text{focal axis}}$. The line through the centre that is perpendicular to the focal axis is called the $\boxed{\text{conjugate axis}}$.

The hyperbola intersects the focal axis at two points called the $\boxed{\text{vertices}}$.



Let the distance between the vertices be denoted by '2a' & the distance between the two foci by '2c'

Define $b = \sqrt{c^2 - a^2}$ — ①

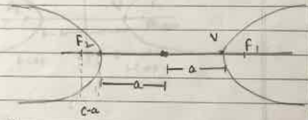
$a \rightarrow$ semifocal axis

$b \rightarrow$ semi conjugate axis

(Note that there are nos., not geometric axes)

Let V be one of the vertices, then the distance from V to the farther focus minus the distance from V to closer focus is

$$\underbrace{[(c-a)+2a]}_{\text{Distance from farther focus } F_2} - (c-a) = 2a \quad - ②$$



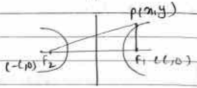
(Note: V lies on the hyperbola)

∴ for all the points on the hyperbola, the distance to farther focus minus the distance to the closer focus is $\boxed{2a}$ (from ②)

<u>Equations of Hyperbola:</u> Let the centre of the hyperbola be origin, x-axis be along the focal axis & y axis along the conjugate axis.

$\therefore$ coordinates of foci are – $F_1 : (c,0)$, $F_2 : (-c,0)$
vertices $V_1(a,0)$, $V_2 : (-a,0)$
Let $P(x,y)$ be any pt on the hyperbola

Now $PF_2 - PF_1 = 2a$



$$\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} = 2a$$

$$\Rightarrow (x+c)^2+y^2 = 4a^2 + (x-c)^2+y^2 + 4a\sqrt{(x-c)^2+y^2}$$
$$\Rightarrow 4cx = 4a^2 + 4a\sqrt{(x-c)^2+y^2}$$
$$\Rightarrow cx - a^2 = a\sqrt{(x-c)^2+y^2}$$
$$\Rightarrow c^2x^2 + a^4 - 2a^2cx = a^2(x-c)^2 + a^2y^2$$
$$\Rightarrow c^2x^2 + a^4 - 2a^2cx = a^2x^2 + a^2c^2 - 2a^2cx + a^2y^2$$
$$\Rightarrow (c^2-a^2)x^2 - a^2y^2 = a^2(c^2-a^2)$$
$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1 \quad (\text{from } ①)$$
$$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad — ③$$

Rewrite this eq^n as
$$y^2 = \frac{b^2}{a^2}(x^2-a^2)$$
$$\Rightarrow y = \pm \frac{b}{a}\sqrt{x^2-a^2} \quad — ④$$

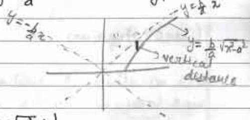
ASYMPTOTES of the hyperbola

Associated with every hyperbola, there is a pair of lines, called the asymptotes (oblique) of the hyperbola. These lines intersect at the center of the hyperbola & have the property that as a point 'P' moves along the hyperbola away from the centre, the vertical distance between P & one of the asymptotes approaches zero

(Asymptotes are lines such that the vertical distance between the curve & lines tends to zero).

→ We shall show that the lines $\boxed{y = \pm \frac{b}{a} x}$ are the two asymptotes to the hyperbola

In the first quadrant, consider the distance between the line $y = \frac{b}{a}x$ & the hyperbola



$\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}$

$\therefore \lim_{x \to \infty} \frac{b}{a}\left[x - \sqrt{x^2 - a^2}\right]$

$= \lim_{x \to \infty} \frac{b}{a} \frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}}$

$= \lim_{x \to \infty} \frac{b}{a} \frac{a^2}{x + \sqrt{x^2 - a^2}} = \lim_{x \to \infty} \frac{ab}{x + \sqrt{x^2 - a^2}} = 0$

Similarly, considering the rest of the quadrant, we have

$y = \pm \frac{b}{a} x$ are two asymptotes to the hyperbola.

A quick way to find asymptotes to the hyperbola:

For the eqn in the standard form, write 0 in place of 1 & solve i.e. the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are given by

$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}$ $\Rightarrow y^2 = \frac{b^2}{a^2}x^2 \Rightarrow \boxed{y = \pm \frac{b}{a} x}$

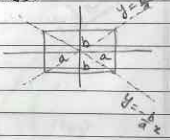
Sketching Hyperbolas:

① Determine whether the focal axis is on the x-axis or the y-axis.
— The focal axis is along the x-axis when the minus sign precedes the y^2 term, and it is along the y-axis when the minus sign precedes the x^2 term

② Let the given eq^n be of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

then, the focal axis is along x-axis
(For the eq^n $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the x-axis is ~~to~~ ~~tent~~ taken along conjugate axis)

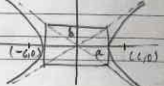
③ Draw a box extending 'a' units on either side of the centre along the focal axis & b units on either side of the centre along the conjugate axis).



(Note the eq^n of the diagonals in this case :- $y = \pm \frac{b}{a}x$, which are asymptotes to the hyperbola)

④ Draw the asymptotes along the diagonals of the box. (we dotted line)

⑤ Using the box & the asymptotes as a guide, sketch the graph



eg Sketch $\frac{x^2}{4} - \frac{y^2}{9} = 1$

① Since minus sign preceeds the y^2 term, the focal axis is taken along x-axis.

$a^2 = 4, \quad b^2 = 9$

$\Rightarrow a = 2, \; b = 3$ (∵ they are +ve)

② Co-ord of vertices are $(2,0)$ & $(-2,0)$

③ $c = \sqrt{a^2+b^2} = \sqrt{4+9} = \sqrt{13} \approx 3.6$

∴ co-ord of focus are $(3.6, 0)$ & $(-3.6, 0)$

④ Asymptotes are given by

$$\frac{x^2}{4} - \frac{y^2}{9} = 0$$

$$\Rightarrow y = \pm \frac{3}{2} x$$

⑤ Draw a box extending 2 units along x-axis on each sides of the origin & 3 units on each sides of the origin along the y-axis.

⑥ Draw asymptotes along the diagonals of the box.

⑦ Sketch the hyperbola