Uniform convergence:

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a f^n $f: A_0 \to \mathbb{R}$ if for each $\epsilon > 0$, $\exists$ a natural no. $K(\epsilon)$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in A_0 \quad \forall n \geq K(\epsilon)$$ $\boxed{\text{—(1)}}$

(Note that the choice of K depends upon 'ϵ' only & not on x.

We say that (f_n) converges uniformly to f on A_0. & we write

$$f_n \rightrightarrows f \text{ on } A_0$$

or $$f_n(x) \rightrightarrows f(x) \text{ for } x \in A_0.$$

Unif conv. $\Rightarrow$ Pt. wise cgt.
conv. is not true

Infact, if a seq^n of f^n is not pt. wise cgt, then, it can not possibly be unif cgt.

Non uniform convergence criterion:

lemma: A seq^n (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a f^n, $f: A_0 \to \mathbb{R}$ iff for some $\epsilon_0 > 0$, there is a subseq^n (f_{n_k}) of (f_n) & a seq^n (x_k) in A_0 s.t.

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \quad \forall k \in \mathbb{N}$$

Pf. let (f_n) does not conv. unif to f.
$\therefore$ $\exists$ an $\epsilon_0 > 0$ st. for each natural no. k, $\exists$ a natural no. n_k & $x_k \in A_0$ s.t

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \qquad \forall k \in \mathbb{N}.$$

($\exists$ an $\epsilon_0 > 0$ for which no natural no. k exists.
i.e. $\exists$ an $\epsilon_0 > 0$ st. for each natural no. k, (I) doesnot hold
i.e. $\exists$ an $\epsilon_0 > 0$ st. for each natural no k, $\exists$ a natural no. n_k & $x_k \in A_0$ st.

$$|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0 \;)$$

eg1 Show that $f_n(x) = \frac{x}{n}$ $\forall x \in \mathbb{R}$, for each n, does not conv. unif to 0.

Soln let $\epsilon_0 = \frac{1}{2}$ let $n_k = k$

$x_k = k$

$\therefore$ $f_{n_k}(x_k) = 1$

$$|f_{n_k}(x_k) - f(x_k)| = 1 \nless \epsilon_0$$

eg2 $g_n(x) = x^n$ $\qquad g(x) = \begin{cases} 0 & -1 < x < 1 \\ 1 & x = 1 \end{cases}$

let $\epsilon_0 = \frac{1}{4}$

$n_k = k$ & $x_k = \left(\frac{1}{2}\right)^{1/k}$

$$|g_{n_k}(x_k) - g(x_k)| = \left|\left(\left(\frac{1}{2}\right)^{1/k}\right)^k - 0\right| = \frac{1}{2} \nless \epsilon_0.$$

eg 3 $\quad h_n(x) = \frac{x^2+nx}{n}$ $\qquad h(x) = x.$

$n_k = k, \quad x_k = -k.$

$h_{n_k}(x_k) = \frac{k^2-k^2}{k} = 0.$

$\therefore \quad |h_{n_k}(x_k) - h(x_k)| = k$

Uniform norm:

Def^n Let $A \subseteq \mathbb{R}$, let $\phi : A \to \mathbb{R}$ be a bounded f^n on A. We define uniform norm of ϕ on A by

$$\|\phi\|_A = \sup\{|\phi(x)| : x \in A\}$$

For $\epsilon > 0$,

$$\|\phi\|_A \le \epsilon \iff |\phi(x)| \le \epsilon \quad \forall x \in A.$$

Lemma: A seq^n (f_n) of bounded f^n on $A \subseteq \mathbb{R}$ converge uniformly on A to f iff
$\|f_n - f\|_A \to 0.$

Pf: Let $f_n \rightrightarrows f$
Given any $\epsilon > 0$, $\exists$ $K(\epsilon)$ s.t.
$|f_n(x) - f(x)| < \epsilon \quad \forall n \ge K. \quad \forall x \in A.$

$\Rightarrow \quad \sup_x |f_n(x) - f(x)| \le \epsilon \quad \forall n \ge K$

$\Rightarrow \quad \|f_n - f\|_A \le \epsilon \quad \forall n \ge K$

$\Rightarrow \quad \|f_n - f\|_A \to 0$
$(\because \epsilon > 0$ is arb$)$

<u>Conversely</u>: Let $\|f_n - f\|_A \to 0$

T.S.T. $f_n \rightrightarrows f$

Let $\epsilon > 0$ be any no.

$\therefore \exists$ $K \in \mathbb{N}$ s.t.

$$\|f_n - f\|_A < \epsilon \quad \forall n \geq K$$

$$\Rightarrow \sup_{x \in A} |f_n(x) - f(x)| < \epsilon \quad \forall n \geq K$$

$$\Rightarrow |f_n(x) - f(x)| < \epsilon \quad \forall n \geq K, \ \forall x \in A$$

$$\Rightarrow f_n \rightrightarrows f.$$

$\boxed{\text{Note}}$ This lemma can be applied only to those f^n, for which, $f_n(x) - f(x)$, is a bdd f^n.

$$f_n(x) = \frac{x}{n} \qquad [0,1] \qquad \text{(Note the diff in the domain)}$$

We shall show that $(f_n(x))$ ~~doesnot~~ converges uniformly on $[0,1]$.

$$f(x) = \lim f_n(x) = 0 \qquad \forall x \in [0,1]$$

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\right|$$

$$\sup |f_n(x) - f(x)| = \sup_{[0,1]} \left|\frac{x}{n}\right| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$\therefore f_n \rightrightarrows f$ on $[0,1]$.

eg2 $g_n(x) = x^n$ on $[0,1]$

$$g(x) = \begin{cases} 0 & x \in [0,1[\\ 1 & x=1 \end{cases}$$

$\therefore$ $\sup|g_n(x) - g(x)| = \sup \begin{cases} |x^n - 0| & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$

$= \sup|x^n|$

$= 1$

$\not\to 0$

$\therefore$ $g_n \not\xrightarrow{u} g$

eg3 $G_n(x) = x^n(1-x)$ $[0,1]$

$G(x) = \lim_{n\to\infty} G_n(x) = 0$ $\forall x$ $(G_n(1) = 0)$

$|G_n(x) - G(x)| = |x^n(1-x)| = x^n(1-x)$ (Note it is +ve in $[0,1]$.

To find sup of $x^n(1-x)$, we shall use max min method

let $f(x) = x^n(1-x)$ on $[0,1]$. (Since sup needs to find on $[0,1]$ we are writing it as a f^n of x)

$f'(x) = n x^{n-1} - (n+1)x^n$

$f'(x) = 0 \Rightarrow x^{n-1}[n - (n+1)x] = 0$

$\Rightarrow x = 0$ or $x = \frac{n}{n+1}$

Consider $x = \frac{n}{n+1} \in (0,1)$ (Note for $x=0$ $f(x)=0$)

$f''(x) = n(n-1)x^{n-2} - n(n+1)x^{n-1}$

$= n x^{n-2}[n-1 - (n+1)x]$

$f''\left(\frac{n}{n+1}\right) < 0$ $\therefore$ $x = \frac{n}{n+1}$ is a pt of max.

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$\therefore \sup |G_n(x) - G(x)| = \left(\frac{n}{n+1}\right)^n \left[1 - \frac{n}{n+1}\right]$

$= \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$

$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{n+1}$

$\to \frac{1}{e} \cdot 0 = 0$

$\therefore \|G_n - G\|_A \to 0$

$\Rightarrow G_n \rightrightarrows G.$

Cauchy Criterion for Uniform Convergence'

Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$.

Then (f_n) converges uniformly on A to a bounded f^n f' iff for each $\epsilon > 0$, $\exists\ H(\epsilon) \in \mathbb{N}$ st.

$\|f_m - f_n\|_A \le \epsilon \quad \forall n, m \ge H(\epsilon).$

Pf: Let $f_n \rightrightarrows f$ on A.

Let $\epsilon > 0$. Then $\exists\ H \in \mathbb{N}$ st.

$\|f_n - f\|_A \le \epsilon/2 \quad \forall n \ge H$

$\Rightarrow |f_n(x) - f(x)| < \epsilon/2 \quad \forall n \ge H \quad \forall x \in A.$

Let $m, n \ge H$

$|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)|$

$\le \epsilon/2 + \epsilon/2 = \epsilon \quad \forall x$

$\Rightarrow \sup |\quad| \le \epsilon$

$\Rightarrow \|f_m - f_n\|_A \le \epsilon \quad \forall n, m \ge H$

Conv. T.S.T. (f_n) is unif cgt. (∵ The condition is given)

Let $\epsilon > 0$. Then $\exists\, H \in \mathbb{N}$ s.t.

$$\|f_m - f_n\|_A < \epsilon \quad \forall n, m \geq H.$$

$$\Rightarrow \sup_x |f_m(x) - f_n(x)| \leq \epsilon \quad \forall n, m \geq H.$$

$$\Rightarrow |f_m(x) - f_n(x)| \leq \epsilon \quad \forall n, m \geq H \quad \forall x \in A. \quad — (*)$$

$\Rightarrow (f_n(x))$ is a Cauchy seq^n of real no. (for each $x \in A$).

$\Rightarrow (f_n(x))$ conv. in $\mathbb{R}$ (for each $x \in A$)

Let $f_n(x) \to f(x)$ for each $x \in A$

(Note that this way we have defined the f^{ing} f on A).

Define $f: A \to \mathbb{R}$

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \text{for each } x \in A.$$

Let $n \to \infty$ in $(*)$ inequality

$$|f_m(x) - f(x)| \leq \epsilon \quad \forall m \geq H \quad \text{for each } x \in A.$$

$\Rightarrow f_n \rightrightarrows f$ on A (Note that H is independent of choice of x).

Assignment: Complete Ex 8.1 from Bartle & Sherbert