Study Material(II)

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1 Linear Independence

In last class we studied about span of a set and Simplified Span method to find a simplified form for span(S) (S is a finite subset of $\mathbb{R}^n$).

Now in this section, we will explore the concept of linearly independent set of vectors and examine methods for determining whether or not a given set of vector is linearly independent. We will also see the connections between the span and linearly Independence.

1.1 Definition

Let $S = \{v_1, v_2, ..., v_n\}$ be a non empty subset of a vector space V. Then S is linearly dependent if and only if there exist real number $c_1, c_2, ..., c_n$ not all zero such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$. That is, S is linearly dependent if and only if the zero vector can be expressed as a nontrivial linear combination of the vectors in S.

S is linearly independent if and only if it is not linearly dependent. In other words, S is linearly independent if the only linear combination of the vectors of S that equals 0 is the trivial linear combination (i.e. all coefficients = 0)

The empty set, $\{\}$, is linearly independent.

Theorem 1 *A set $S = \{v\}$ containing exactly one element is linearly dependent if and only if $v = 0$. Equivalently, $S = \{v\}$ is L.I. if and only if $v \neq 0$.*

Proof Try to prove it yourself.

Theorem 2 *A set $S = \{v_1, v_2\}$ containing exactly two vectors is L.D. if and only if at least one of the vector is a scalar multiple of the other. Equivalently, a set $S = \{v_1, v_2\}$ is L.I. if and only if neither of the vectors is a scalar multiple of the others.*

Proof Let us suppose that S is L.D. set with two elements. then there exist real numbers c_1, c_2 not both zero such that

$$c_1 v_1 + c_2 v_2 = 0 \tag{1}$$

If $c_1 \neq 0$, this implies that $v_1 = -\frac{c_2}{c_1} v_2$. That is v_1 is scalar multiple of v_2. Similarly if $c_2 \neq 0$ we see that v_2 is a scalar multiple of v_1.

Conversely suppose that $v_1 = kv_2$ for some $k \in \mathbb{R}$. Then

$$1v_1 + (-k)v_2 = 0$$

gives a non-triviallinear combination of v_1 and v_2 that equals the zero vector. Thus, by the definition of the set S is linearly dependent.Hence proved.

Theorem 3 *Any finite subset of a vector space contains the zero vector is linearly dependent.*

Proof Let S be a finite subset of a vector space V containing the zero vector. Case 1. When $S = \{0\}$, That is S is containing only zero vector, then by Theorem 1, S is linearly dependent. Case 2. When $S = \{v_1, v_2, ..., v_n\}$ contains at least two distinct vectors with one of them 0(say $v_1 = 0$), then

$$1v_1 + 0v_2 + ... + 0v_n = 1.0 + 0 + 0 + ... + 0 = 0$$

We have thus expressed the zero vector as a non-trivial combination of the vectors in S. Hence, by the definition, S is linearly dependent. Hence, by case 1 and case 2 S is linearly dependent iff it contains zero vector.

Example Examine whether the subset $S = \{[1, -1, 0, 2], [0, -2, 1, 0], [2, 0, -1, 1]$ of $\mathbb{R}^4$ is linearly independent.

Solution By definition of L.I of S we proceed by assuming that

$$a[1, -1, 0, 2] + b[0, -2, 1, 0] + c[2, 0, -1, 1] = [0, 0, 0, 0]$$

we will show that a, b and c all are zero. Equating the coordinates on each side lead to the following homogeneous system:

$$a + 0b + 2c = 0$$

$$-a - 2b + 0c = 0$$

$$0a + b - c = 0$$

$$2a + 0b + c = 0$$

The augmented matrix for the above system is:
$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right]$ which row reduce to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ From row reduced echoln form we have $a = 0, b = 0, c = 0$. Thus, S is linearly independent.

Test for Linear Independence using Row Reduction (Independence Test Method) Let S be a finite nonempty set of vectors in $\mathbb{R}^n$. To determine whether S is linearly independent, perform the following steps:

Step 1: Create the matrix A whose columns are the vectors in S.

Step 2: Find B, the reduced row echelon form of A.

Step 3: If there is pivot in every column of B, then S is linearly independent otherwise, S is linearly dependent.

Example Use the Independence Test method to determine which of the following sets of vectors are linearly independent.
(a) $S_1 = \{[3, 1, -1], [-5, -2, 2], [2, 2, -1]\}$
(b) $S_2 = \{[2, 5], [3, 7], [4, -9], [-8, 3]\}$
Solution (a) To determine whether S_1 is linearly independent using the Independence Test Method, we first create the matrix

$$A = \begin{bmatrix} 3 & -5 & 2 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

whose columns are the vectors in S_1. We next find the matrix B, the reduced row echelon form of A.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is pivot in every column of B, the set S_1 is linearly independent.
(b) To determine whether S_2 is linearly independent using the Independence Test Method, we first create the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 & -8 \\ 5 & 7 & -9 & 3 \end{bmatrix}$$

whose columns are the vectors in S_2. We next find the matrix B, the reduced row echelon form of A.

$$B = \begin{bmatrix} 1 & 0 & -55 & 65 \\ 0 & 1 & 38 & -46 \end{bmatrix}$$

Since we have no pivot in column 3 and 4 of B, the set S_2 is linearly dependent.
Note In last Example, there are more columns than rows in the matrix we row reduced. Hence, there must definitely be some column without a pivot, since each pivot is in a different row. Consequently in such case the original set of vectors must be linearly dependent. Thus we have the following result:

Theorem 4 *(Without Proof) If S is any subset of $\mathbb{R}^n$ containing k distinct vectors, where $k > n$, then S is linearly dependent.*

Example Use the Independence method to determine whether the subset $S = \{x^2 + x + 1, x^2 - 1, x^2 + 1\}$ of P_2 is linearly independent.
Solution First we convert the polynomial in S into vectors in $\mathbb{R}^3$

$$x^2 + x + 1 \to [1, 1, 1], x^2 - 1 \to [1, 0, -1], x^2 + 1 \to [1, 0, 1]$$

Now we use the Independence Test Method on the set $T = \{[1, 1, 1], [1, 0, -1], [1, 0, 1]\}$ of vectors converted from the polynomial in S. we create the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

whose columns are the vectors in the set T. We now reduce the matrix A to obtain

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since there is pivot in every column of B, the set T and hence the set S is linearly independent.
Exercise Show that the following is a linearly independent:

$$\left\{ \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 6 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} -11 & 3 \\ -2 & 2 \end{bmatrix} \right\}$$

Theorem 5 ***Alternative Characterization of Linear Independence*** *Suppose S is a finite set of vectors having at least two vectors. Then S is linearly dependent if and only if some vector in S can be expressed as a linear combination of the other vectors in S.*

Proof First we assume that S is linearly dependent. Then there exsit real number $c_1, c_2, ..., c_n$ such that

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$$

With $a_i \neq 0$ for some i. This implies that

$$v_i = \left(-\frac{a_1}{a_i}\right) v_1 + \left(-\frac{a_{i-1}}{a_i}\right) v_{i-1} + ... + \left(-\frac{a_n}{a_i}\right) v_n$$

Which express v_i as a linear combination of the other vectors in S.
Conversely, We assume that there is a vector v_i in S that is a linear combination of the other vectors in S. Without any loss of generality, we assume that $v_i = v_1$. Thus, there are real numbers $c_1, c_2, ..., c_n$ such that

$$v_1 = c_2 + v_2 + c_3 v_3 + ... + c_n v_n$$

$$\implies -v_1 + c_2 v_2 + c_3 v_3 + ... + c_n v_n = 0$$

Thus, if we let $c_1 = -1$, then $c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$, with $c_1 \neq 0$. Therefore, by the definition, the set S is linearly dependent, Hence proved.

Exercise Try Example 10 and 11 of Andrilli.
The following Corollary gives another characterization of linear dependence using the concept of Span.
corollary A set S in a vector space V is linearly dependent if and only if there is some vector $v \in S$ such that $v \in span(S - \{v\}$. Equivalently, S is linearly independent if and only if there is no vector $V \in S$ such that $v \in span(S - \{v\}$.

Another useful characterization of linear independence is the following:

Theorem 6 *Let $S = \{v_1, v_2, ..., v_n\}$ be a non-empty subset of a vector space V. Then S is linearly independent if and only if*

1. $v_1 \neq 0$; and

2. for each $k, 2 \leq k \leq n, v_k \notin span\{v_1, v_2, ..., v_n\}$

Proof Let us assume that S is linearly independent then it cannot contain zero vector. Therefore, all the vectors are nonzero. Hence, $v_1 =\neq 0$. This proves part 1. Now we need to show part 2, for this let us suppose that there is a $k, 2 \leq k \leq n$, such that

$$v_k \in span(\{v_1, v_2, ..., v_{k-1}\})$$

Then there exist real numbers $a_1, a_2, ..., a_{k-1}$ such that

$$v_k = a_1 v_1 + a_2 v_2 + ... + a_{k-1} v_{k-1}$$

$$\implies a_1 v_1 + a_2 v_2 + ... + a_{k-1} v_{k-1} + (-1) v_k = 0$$

Now by letting $a_k = -1$ and $a_{k+1} = ... = a_n = 0$, we get

$$a_1 v_1 + a_2 v_2 + ... + a_k v_k + ... + a_n v_n = 0$$

Since $a_k \neq 0$, this shows that the set $S = \{v_1, v_2, ..., v_n\}$ is linearly dependent, which contradict our assumption that S is linearly independent. This implies that $v_k \notin span\{v - 1, v_2, ..., v_{k-1}\}$
Conversely, we assume that a non-empty subset $S = \{v_1, v_2, ..., v_n\}$ of a vector space V satisfies conditions (1) and (2). To prove that S is linearly independent , we must show that for any set of real numbers $a_1, a_2, ..., a_n$, the equation

$$a_1 v_1 + a_2 v_2 + ... + a_n v_n \implies a_1 = a_2 = ... = a_n = 0$$

Notice that if $a_1, a_2, ..., a_n$ are all zero , then a_1 must be zero (because $v_1 \neq 0$), thus completing the proof in this case. On the other hand, if not all of $a_2, a_3, ..., a_n$ are zero, we let k to be the largest element of $\{2, 3, ..., n\}$ such that $a_k \neq 0$. Then

$$v_k = \left(-\frac{a_1}{a_k}\right) v_1 - ... \left(-\frac{a_{k-1}}{a_k}\right) v_{k-1}$$

which shows that $v_k \in span(\{v_1, v_2, ..., v_{k-1})$, Contradicting condition (2). Hence our assumption is wrong. therefore all coefficients must be zero. this implies that S is linearly independent.

Theorem 7 *Let $S = \{v_1, v_2, ..., v_n\}$ be a non-empty finite subset of a vector space V. Then S is linearly independent if and only if every $v \in span(S)$ can be expresses uniquely as a linear combination of the elements of S.*

Proof Let $S = \{v_1, v_2, ..., v_n\}$. Suppose first that S is linearly independent. Assume that $v \in span(S)$ can be expressed both as

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n \quad , \quad v = b_1v_1 + b_2v_2 + ... + b_n$$

In order to show that the linear combination for v is unique, we need to prove that $a_i = b_i$ for all i. But

$$0 = v-v = (a_1v_1+a_2v_2+...+a_nv_n)-(b_1v_1+b_2v_2+...+b_n) = (a_1-b_1)v_1+(a_2-b_2)v_2+...+(a_n-b_n)v_n$$

Since S is linearly independent set, each $a_i - b_i = 0$, by the definition of linear independence , we have $a_i = b_i, \forall i$. Hence every vector $v \in span(S)$ cab be expressed uniquely as a linear combination of the elements of S.

Conversely, Assume every vector in $span(S)$ can be uniquely expressed as a linear combination of elements of S. Since $0 \in span(S)$, there is exactly one linear combination $a_1v_1 + a_2v_2 + ... + a_nv_n$ of elements of S that equals 0. But the act that $0 = 0v_1 + ... + 0v_n$ together with the uniqueness of expression for 0 means $a_1, a_2, ..., a_n$ are all zero. Thus by the definition of linear combination, S is linearly independent. Hence, Proved.

Exercise Try to prove Example 12(Page no. 246) of Andrilli.

Linear Independence of Infinite Sets

We now extend the definition of linear independence and linear dependence to Infinite sets.

Definition Linear Independence of Infinite

An infinite subset S of a vector space V is said to be **linearly dependent** if there is some finite subset T of S that is linearly dependent. The Set S is said to be **linearly independent** if S is nto linearly dependent. Equivalently, S is linearly independent if every finite subset of S is linearly independent.

Example Consider the subset S of M_{22} consiting of all non singular 2×2. We will show that S is linearly independent.

Let $T = \{I_2, 2I_2\}$, a subset of S. Clearly, $2I_2$ is multiple of I_2, i.e. second element of T is scalar multiple of the first element of T. Therefore, T is a linearly dependent set. Hence, S is linearly dependent, since one of its finite subset is linearly dependent.

Example Let $S = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, ...\}$, S is an infinite subset of $\mathbb{P}$.

Now we Claim that S is linearly independent. To prove our claim, we must prove that every finite subset of S is linearly independent. Let

$$T\{p_1, p_2, ...p_n\}$$

be ant finite subset of S. Without loss of generality, we can assume that

$$deg(p_1) < deg(p_2) < ... < deg(p_n)$$

Now suppose that

$$a_1p_1 + a_2p_2 + ... + a_np_n = 0$$

We need to show that $a_1 = a_2 = ... = 0$. we must prove this by contradiction.

Suppose $a_i \neq 0$ for some i. Let a_k be the last nonzero coefficient. Then,

$$a_1p_1 + a_2p_2 + ... + a_kp_k = 0, with, a_k \neq 0$$

Hence $a_k \neq 0$. Then

$$p_k = \left(-\frac{a_1}{a_k}\right)p_1 - \left(-\frac{a_2}{a_k}\right)p_2 - ... \left(-\frac{a_{k-1}}{a_k}\right)p_{k-1}$$

. Because all the the degrees of the polynomial in T are different and they were listed in order of increasing degree, this equation expressed expresses p_k as a linear combination of polynomial whose degrees are lower than that of p_k. This is not possible. Hence our assumption is wrong. Thus, $a_1 = a_2 = ... = a_n = 0$. Therefore, S is linearly independent. Hence, proved.

The next theorem is generalization of Theorem 7 to include both finite and infinite sets.

Theorem 8 *Let S be a nonempty susbet of a vector space V. Then S is linearly independent if and only if every vector $v \in span(S)$ can be expressed uniquely as a finite linear combination of the elements of S, if the terms with zero coefficients are ignored.*