Study Material(III)

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1 BASIS FOR A VECTOR SPACE

Consider the set $\{(1,0,0),(0,1,0)\}$ and $\{(1,0,0),(0,1,0),(0,0,1)\}$ these two sets are L.I. However the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is something special because, it spans $\mathbb{R}^3$.

Definition A subset $B=\{v_1,v_2,...,v_n\}$ of a vector space V. Then B is called basis for V if

1. B is L.I.

2. $span(B)=V$

Recall that condition (2.) says that every element of V can be expressed as a Linear Combination of elements of B.

Examples 1. The set $\{e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)\}$ is basis of V. The basis $\{e_1,e_2,e_3\}$ is called the standard basis for $\mathbb{R}^3$.

2. The set $\{(1,1,0),(0,-1,1),(1,0,1)\}$ is L.D. Hence it is not a basis for $\mathbb{R}^3$. Note the set can't span $\mathbb{R}^3$.

3. The set $\{(1,1,1),(1,1,0),(1,0,0)\}$ is a basis for $\mathbb{R}^3$. This is different from the standard basis.

Exercise 1. Verfiy that $\{e_1=[1,0,...,],e_2=[0,1,...,0],...,e_n=[0,0,...,1]\}$ is a basis for the vector space $\mathbb{R}^n$.

Example 4. The empty set ϕ is a basis for the trivial vector space $V=\{0\}$, because by definition the empty set is linearly independent and it also spans the trivial vector space.

Exercise 2. Verify that the set

$$B=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$$

forms a basis for matrix M_{23} and it is standard basis for M_{23}.
Note More generally, the standard basis for M_{mn} is defined to be the set consisting of $m \cdot n$ matrices:

$$[\psi_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$$

where ψ_{ij} is the $m \times n$ matrix with 1 in the (i, j) position and zeros elsewhere.

Exercise 3. Verify that the subset $B = \{1, x, x^2, x^3\}$ of P_3 is a basis for P_3.
Note The set $\{1, x, x^2, ..., x^n\}$ containing $n+1$ elements is called Standard Basis for P_n.

Theorem 1 *A set $B = \{v_1, v_2, ..., v_n\}$ of vectors of V is a basis for V iff every $v \in V$ can be written uniquely in the form*

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

Where $a_1, a_2, ..., a_n \in \mathbb{R}$.

Proof We first suppose that B is a basis for v.
We need to show that any $v \in V$ can be written as a linear combination of vectors from B and this representation (Linear Combination) is unique.

Let $v \in V$, Since B spans V, then there exist real numbers $a_1, a_2, ..., a_n$ such that

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n \tag{1}$$

Now we show that the representation in (1) is unique, Suppose that $b_1, b_2, ..., b_n$ are scalar such that e also have

$$v = b_1v_1 + b_2v_2 + ... + b_nv_n \tag{2}$$

Now we subtract (2) form (1), we get

$$0 = v - v = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + ... + a_nv_n$$

Since $B = \{v_1, v_2, ..., v_n\}$ is a basis for V. Hence B is linearly independent. Therefore $a_i - b_i = 0$. This implies that $a_1 = b_1, a_2 = b_2, ..., a_n = b_n$. This shows that representation in (1) is unique. This completing the proof in one direction.

Conversely, Assume every $v \in V$ can be written uniquely in the form given by (1). Clearly by the definition of spanning set, $B = \{v_1, v_2, ..., v_n\}$ spans V. Now we need to show that B is linearly independent. Suppose $a_1, a_2, ..., a_n \in \mathbb{R}$ are such that

$$0 = a_1v_1 + a_2v_2 + ... + a_nv_n$$

But by the fact that $0 = 0v_1 + 0v_2 + ... + 0v_n = a_1v_1 + a_2v_2 + ... + a_nv_n$ then by uniqueness we have $a_1 = a_2 =, ..., a_n = 0$. Thus the set $B = \{v_1, v_2, ..., v_n\}$ is linearly independent and hence is a basis for V.

2 DIMENSION OF A VECTOR SPACE

In this section we will show that if a vector space has one basis that is finite, then all bases for V are finite, and have the same size.

Note $|S|$ denotes the number of elements in a set S.

Lemma (Without Proof) Let S and T be subsets of a vector space V such that S spans V, S is finite, and T is linearly independent. Then T is finite and $|T| \leq |S|$.

Note This Lemma says that if a vector space contains a finite set that span it, then every linearly independent subset of V is also finite and moreover the size of every linearly independent set of vectors is less than or equal to the size of every spanning set of vectors.

Note The next two example show how the above lemma can be used to show, without any computation of that certain sets are not linearly independent and that certain sets do not span a given vector space.

Example 5 Let $T = \{[1,4,3],[2,-7,6],[5,5,-5],[0,3,19]\}$, a subset of $\mathbb{R}^3$. show that T is not linearly independent in $\mathbb{R}^3$.

Solution Since the set $S = \{[1,0,0].[0,1,0],[0,0,1]\}$ is a spanning set for $\mathbb{R}^3$ containing three elements. Thus by previous Lemma no set containing more than three elements in $\mathbb{R}^3$ is linearly independent. In particular the set T is not linearly independent.

Example 6 Let $S = \{[1,2,3,-5],[4,5,8,3],[9,6,7,-1]\}$, a subset of $\mathbb{R}^4$. Show that S does not span $\mathbb{R}^4$.

Solution Since the set $T = \{[1,0,0,0].[0,1,0,0],[0,0,1,0],[0,0,0,1]\}$ containing 4 elements is linearly independent in $\mathbb{R}^4$, so no subset of $\mathbb{R}^4$ of size less than 4 spans $\mathbb{R}^4$. In particular, the set S cannot span $\mathbb{R}^4$.

Theorem 2 *Let V be a vector space, and let B_1 and B_2 be bases for V such that B_1 has finitely many elements. Then B_2 also has finitely many elements, and $|B_1| = |B_2|$.*

Proof Since B_1 and B_2 are bases for V, B_1 spans V and B_2 is linearly independent. Hence by previous Lemma, B_2 has finitely many elements and $|B_2| \leq |B_1|$. Now since B_2 is finite, we can interchange the roles of $|B_1|$ and $|B_2|$ to deduce that $|B_1| \leq |B_2|$. Thus, $|B_1| = |B_2|$.
Note It follows from this theorem that if a vector space V has one basis containing a finite number of elements, then every basis for V is finite, and all bases for V have the same size.

Note We often say that $\mathbb{R}^1$ is one dimension, $\mathbb{R}^2$ is two dimensional and $\mathbb{R}^3$ is three dimensional. Note that $\{1\}$ is a basis for $\mathbb{R}$, $\{[1,0],[0,1]\}$ is a basis for $\mathbb{R}^2$ and $\{e_1,e_2,e_3\}$ is a basis for $\mathbb{R}^3$. Similarly, $\{e_1,e_2,...,e_n\}$ is a basis for $\mathbb{R}^n$ which is called $n-dimensional$ vector space. We say that the concept of dimension is related to the number of elements in a basis. We define this concept for any vector space.

DEFINITION :- A vector space V is said to be finite dimensional if it has a basis B containing a finite number of elements. The dimension of a finite dimensional vector space V, denoted by $dim(V)$, is the number of elements in any basis for V.

A vector space V is called infinite dimensional if it is not finite dimensional. Equivalently, V is infinite dimensional if it has not finite basis.

Example 7 The set $\{e_1=(1,0,0),e_2=(0,1,0),e_3=(0,0,1)\}$ is basis of V. The basis $\{e_1,e_2,e_3\}$ is called the standard basis for $\mathbb{R}^3$. Therefore, $dim(\mathbb{R}^3)=3$. More generally, the space $\mathbb{R}^n$ has basis $\{e_1,e_2,...,e_n\}$, containing n elements. Therefore, $dim(\mathbb{R}^n)=n$.

Example 8 The subset $B=\{1,x,x^2,x^3\}$ of P_3 is a basis for P_3. it containing 4 elements. Therefore, $dim(P_3)=4$. Similarly, the standard basis for P_n is $\{1,x,x^2,...,x^n\}$. This bases has $n+1$ elements, therefore $dim(P_n)=n+1$.

Example 9 The standard basis for M_{23} is :

$$B=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}$$

This basis has 6 elements, therefore $dim(M_{23}=6$. In general the standard basis for M_{mn} is the set $[\psi_{ij}:1\le i\le m,1\le j\le n]$ where ψ_{ij} is the $m\times n$ are all zeroes except for the $(i,j)^{th}$ entries which is 1. This set has mn elements, therefore $dim(M_{mn})=mn$.

Example 10. Consider P, the space of all polynomial with coefficient in $\mathbb{R}$. Show that P is infinite dimensional.

Solution We prove this result by contradiction. Let us assume that P has finite dimension, say $dim(P)=n$. Let S be a basis for P containing n elements. Let m be the highest degree of the polynomial which appear in S. Then every polynomial in the span of S has degree at most m. Thus, x^{m+1} is not in the span of S which contradict the fact that S is a basis for P. Hence P is infinite dimensional.

Size of Spanning sets and Linearly Independent Sets

Theorem 3 *Let V be a finite dimensional vector space.*
(1) *Suppose S is a finite subset of V. Then $dim(V) \leq |S|$. Moreover, $|S| = dim(V)$ if and only if S is a basis for V.*

(2) *Suppose T is a linearly independent subset of V. Then T is finite and $|T| \leq dim(V)$. Moreover, $|T| = dim(V)$ if and only if T is a basis for V.*

Proof Let B be a basis for V with $|B| = n$. Then $dim(V) = |B| = n$.
Part (1) Since S is a finite spanning set and B is linearly independent. Then By previous Lemma this implies that $|B| \leq |S|$, and so $dim(V) \leq |S|$

If $|S| = dim(V)$, we prove that S is a basis for V by contradiction. Let us assume that S is not a basis for V, then it is not linearly independent (Because it spans V). So by corollary (from section Linear independence), there exist a vector $v \in S$ such that $span(S_{\{}v\} = span(S) = V$. But then $S - \{v\}$ is a spanning set for V containing fewer than n elements. This is a contradiction to the fact that we have just observed that the size of a spanning set is never less than the dimension.

Finally, suppose S is a basis for V. By Theorem 2, S is finite, and $|S| = dim(V)$ by the definition of dimension.

Part(2) Using B as the spanning set S in Lemma of this section proves that T is finite and $|T| \leq dim(V)$.

If $|T| = dim(v)$, we prove that T is basis for V by contradiction. If T is not basis for V then T does not span V (because it is linearly independent). Therefore, there is a vector $v \in V$ such that $v \notin span(T)$. Then $T \cup \{v\}$ is also linearly independent. But $T \cup n + 1$ elements, contradicting the fact we just proved that size of a linearly independent set is always less than the dimension.

Finally, if T is a basis for V, then $|T| = dim(V)$, by the definition of dimension.

Example 11. Examine whether the subset $S = \{[7, 1, 2, 0], [8, 0, 1, -1], [1, 0, 0, -2]\}$ of $\mathbb{R}^4$ forms a basis for $\mathbb{R}^4$.

Solution The set S is not a basis for $\mathbb{R}^4$, because S cannot span $\mathbb{R}^4$. For if S spans $\mathbb{R}^4$, then by part (a) of Theorem 3

$$4 = dim(\mathbb{R}^4 \leq |S| = 3$$

Which is contradiction.

Example 12. show that the set $S + \{[2, 3], [5, 1]\}$ is a basis for $\mathbb{R}^2$

Solution The set S is obviously independent in $\mathbb{R}^2$. Note that $dim(\mathbb{R}^2) = 2$. Thus, Theorem 3 implies that the linearly independent set S of size two ia a basis for $\mathbb{R}^2$.

Exercise 4. Prove that the set $S = \{3, 1, -1], [5, 2, -2], [2, 2, -1]\}$ is linearly independent in $\mathbb{R}^3$. Examine whether S forms a basis for $\mathbb{R}^3$.

Exercise 5. Show that the set $B = \{[-1, 2, -3], [3, 1, 4], [2, -1, 6]\}$ is a basis for $\mathbb{R}^3$.

Definition:- (Maximal Linearly Independent Sets and Minimal Spanning Sets)

A subset S of a vector space V is said to be **Maximal linearly independent subset** of V if S is linearly independent and any subset T of V that properly contains S is linearly dependent.

A subset S of a vector space V is said to be **Minimal Spanning Subset** of V if S spans V and no proper subset of S spans V.

Example 13. Let $B = \{[2, 3, 0, -1], [-1, 1, 1, -1]\}$ and $S = \{[1, 4, 1, -2], [-1, 1, 1, -1], [3, 2, -1, 0], [2, 3, 0, -1]\}$.

(a) Show that B is maximal linearly independent subset of S.2

b Calculate $dim(span(S))$.

(c) Does $span(S) = \mathbb{R}^4$? Why or why not ?

Solution(a) The set B is clearly linearly independent, because neither of the vector in B is a multiple of the other (Check it). We now show that B is a maximal linearly independent subset of S by showing that any subset of S that properly contains B is linearly dependent. The only subset of S that properly contains B are

$$T_1 = \{[2, 3, 0, -1], [-1, 1, 1, -1], [1, 4, 1, -2]\}$$

$$T_2 = \{[2, 3, 0, -1], [-1, 1, 1, -1], [3, 2, -1, 0]\}$$

$$T_3 = \{[2, 3, 0, -1], [-1, 1, 1, -1], [1, 4, 1, -2], [3, 2, -1, 0]\} = S$$

By Independence Test Method check that T_1, T_2 and T_3 are linearly dependent.

Since, T_1, T_2 and T_3 are containing B and these are linearly dependent. Therefore B is maximal linearly independent subset of T.

(b) We first use the Simplifeid Span method to find a simplified form for the vectors in $Span(S)$. for this we create the matrix A

$$A = \begin{bmatrix} 1 & 4 & 1 & -2 \\ -1 & 1 & 1 & -1 \\ 3 & 2 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

Whose rows are the vector in S. Now row reduce form for the matrix A is (Check it)

$$C = \begin{bmatrix} 1 & 0 & -\frac{3}{5} & \frac{2}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix C is the reduced row echelon form of A and

$$W = span(S) = span(\{[5, 0, -3, 2], [0, 5, 2, -3]\})$$

The vectors $[5, 0, -3, 2]$ and $[0, 5, 2, -3]$ are obviously linearly independent, because neither of the vectors is multiple of the others. Therefore, the set $B = \{[5, 0, -3, 2], [0, 5, 2, -3]\}$ forms a basis for W. Hence,

$$dim(W) = |B| = 2.$$

(c) $span(S) \neq \mathbb{R}^4$ because $dim(span) = 2 \neq dim(\mathbb{R}^4)$.

Theorem 4 *(Without Proof) Let S be a subset of a vector space V. Then the following statements are equivalent:*
(a) *S is a basis for V.*
(b) *S is minimal spanning subset for V.*
(c) *S is a maximal linearly independent subset of V.*

The next theorem shows an important connection between the dimension of a finite dimensional vector space and it subspace.

Theorem 5 *(Without Proof) Let V be a finite dimensional vector space, and let W be a subspace of V. Then W is also finite dimensional with $dim(W) \leq dim(V)$. Moreover, $dim(W) = dim(V)$ if and only if $W = V$*

Exercise 6. Find a basis and the dimension for the subspace W of $\mathbb{R}^3$ spanned by the set $S = \{[3, 2, 1], [1, 2, 0], [-1, 2, -1]\}$ of three vectors.