Finite Dimensional Vector Spaces

In previous classes, we have discussed about vector spaces, subspace, various elementary properties of vector spaces and understand the concept of a linear combinations of vectors in a genral genral vector space.

We also discussed about spanning set and related theorems and corollary.

In last class we proved the following theorem.

Theorem : 4.5 Span (S) is the minimal subspace containing S.

Statement: Let S be a non-empty subset of a vector space V. Then,

1. $S \subseteq \text{Span}(S)$
2. Span(S) is a subspace of V. (under the Same operation of V).
3. If W is a subspace of V with $S \subseteq W$, then $\text{span}(S) \subseteq W$.
4. Span(S) is the smallest subspace of V containing S.

Corollary: 4.6 Let V be a vector space, and let S_1 and S_2 be subsets of V with $S_1 \subseteq S_2$. Then $\text{span}(S_1) \subseteq \text{span}(S_2)$.

Now, we prove following Result.

Theorem: 4.7 Let S be a non-empty subset of a vector space V. Then S is a subspace of V iff $\text{span}(S) = S$.

Proof: ~~Since~~ first suppose that S is a subspace of V.

We need to show that: $\text{span}(S) = S$
i.e. (i) $\text{span}(S) \subseteq S$
(ii) $\text{span}(S) \supseteq S$

By Theorem 4.5, we know that $\text{span}(S)$ is the smallest subspace of V containing S, so $\text{span}(S) \subseteq S$. Also, for any non empty subset S of V, we always have $S \subseteq \text{span}(S)$.

Therefore, $\text{span}(S) = S$.

Conversely, assume that $\text{span}(S) = S$.
Then, we know S is subspace of V.
Therefore, $\text{span}(S)$ is a subspace of V.

example: Consider $S = \{[1,-1,3,0], [2,-1,8,-1]\}$

Then clearly S is subset of $\mathbb{R}^4$.

Need to find span (S).

We know by Theorem 4.5 span (S) is the smallest subspace of $\mathbb{R}^4$ containing S.

By definition of spanning set. We have

$$a[1,-1,3,0] + b[2,-1,8,-1] = [a+2b, -a-b, 3a+8b, -b].$$

$\therefore$

$$\text{Span}(S) = \{[a+2b, -a-b, 3a+8b, -b] \mid a, b \in \mathbb{R}\}.$$

Exercise: is $S = \{[1,0,1], [1,1,0], [0,1,1]\} \subseteq \mathbb{R}^3$ generates $\mathbb{R}^3$?

* Simplifying Span (S) using Row Reduction.

Our next goal is to find a simplified form for the vectors in the span of a given set S.

• Method for Simplifying Span (S) using Row Reduction.

Suppose that S is a finite subset of $\mathbb{R}^n$ containing k vectors, where $k \geq 2$

Step. 1 form a $k \times n$ matrix A by using the vectors in S as the rows of A. (Thus, Span (S) is the row space of A).

Step. 2 Find C, the reduced row echelon form of A.

Step. 3 A simplified form for span(S) is given by the set of all linear combinations of the non-zero rows of C.

Example: Consider the subset

$S = \{[1,4,-1,5], [2,8,5,4], [-1,-4,2,7], [6,24,-1,-20]\}$ of $\mathbb{R}^4$. Use the simplified method to find span(S) in simplified form.

Solution: By definition, span (S) is the set of all vectors of the form

$a[1,4,-1,5] + b[2,8,5,4] + c[-1,-4,2,7] + d[6,24,-1,-20]$ for $a,b,c,d \in \mathbb{R}$.

Step: 1 Form the matrix A by using the vectors in S as the rows of A.

$$A = \begin{bmatrix} 1 & 4 & -1 & -5 \\ 2 & 8 & 5 & 4 \\ -1 & -4 & 2 & 7 \\ 6 & 24 & -1 & -20 \end{bmatrix}$$

Step: 2 Row Reduce A to C.

$$C = \begin{bmatrix} \boxed{1} & 4 & 0 & -3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step: 3 A simplified form for span(S) is given by the ~~set~~ of all linear combinations of the non-zero rows of C:

$\text{Span}(C) = \{ a[1,4,0,-3] + b[0,0,1,2] \mid a, b \in \mathbb{R} \}$

$= \{ [a, 4a, b, -3a + 2b] \mid a, b \in \mathbb{R} \}$

Note: for example, that the vector $[3, 12, -2, -13]$ is in span(S) when we put $a = 3, b = -2$. we get $[3, 12, -2, -13]$.

But the vector $[-2, -8, 4, 6]$ is not in span(S) because the following system has no solution.

$[a, 4a, b, -3a + 2b] = [-2, -8, 4, 6]$.

$$\Rightarrow \left.\begin{aligned} a &= -2 \\ 4a &= -8 \\ b &= 4 \\ -3a + 2b &= 6 \end{aligned}\right\}.$$

Homework: (I) Prove that the set
$\{x^2+4x-3,\ 2x^2+x+5,\ 7x-11\}$
does not span P_2.

(2) (a) Let $S = \{[1,-2,-2],\ [3,-5,1],\ [-1,1,5]\}$
show that $[-4,15,-13] \in \text{span}(S)$ by expressing it as a linear combination of the vectors in S.

(b) Prove that the set S in part (a) does not span $\mathbb{R}^3$.

Example: Consider the subset

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $M_{2\times 2}$. Use the simplified span method to find a simplified form for all the vectors in span(S).

Solution: First, we convert the matrices in S into vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow [-1,1,0,0],\quad \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \longrightarrow [0,0,1,-1]$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow [-1,0,0,1].$$

Now, use the simplified span method on these vector.

To do this, we form the matrix A:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Now, row reduce matrix A to obtain the reduced row echelon matrix C, ~~as shown~~.

Thus, the matrix A row reduces to

$$C = \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

Now, we must convert the non-zero rows of C to matrix form:

$$[1, 0, 0, -1] \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[0, 1, 0, -1] \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$ and

$$[0, 0, 1, -1] \rightarrow \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

$$\therefore \text{span}(S) = \left\{ a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Note: Special case: The span of the empty set

Definition: $\text{span}(\{\}) = \{0\}$.

$\{\}$ ↓ empty set

Note: Try to solve the following excersise problems from section 4.3 of Stephen Andrilli and David Hecker.

1, 2, 3, 6, 7, 8, 11.