Study Material (II)

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1 Orthogonal Projection

In previous we discussed about Orthogonal Complement. In this we will present orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$.

Orthogonal Projection Onto A Subspace

Recall from chapter that if x is a nonzero vector in $\mathbb{R}^n$, then every vector y in $\mathbb{R}^n$ can be decomposed as the sum of two component vector, $proj_x y$ and $y - proj_x y$, where the first is parallel to x and the second is orthogonal to x.

The following theorem gives a generalization of Theorem 1.10 (Chapter 1, page no. 26)

Theorem 1 *(Projection Theorem) Let W be a subspace of $\mathbb{R}^n$. Then every vector $v \in \mathbb{R}^n$ can be expressed in a unique way as $w_1 + w_2$, where w_1 W and $w_2 \in W^{\perp}$.*

Proof Let W be a subspace of $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$. We first show that v can be expressed as $w_1 + w_2$, where $w_1 \in W$, $w_2 \in W^{\perp}$. Then we will show that there is a unique pair w_1, w_2 for each v.

Let $\{u_1, u_2, ..., u_k\}$ be an orthogonal basis for W. We expand orthogonal basis $\{u_1, u_2, ..., u_k\}$ to an orthonormal basis $\{u_1, u_2, ..., u_k, u_{k+1}, ..., u_n\}$. Then by Theorem 6.3 (Page No. 399 of Andrilli), $v = (v \cdot u_1)u_1 + ... + (v \cdot u_n)u_n$. Let $w_1 = (v \cdot u_1)u_1 + ... + (v \cdot u_k)u_k$ and $w_2 = (v \cdot u_{k+1})u_{k+1} + ... + (v \cdot u_n)u_n$. Clearly $v = w_1 + w_2$. Also, Theorem 6.12 (Page No. 413) implies that $w_1 \in W$ and $w_2 \in W^{\perp}$.

Finally we want to show uniqueness of decomposition. Suppose that $v = w_1 + w_2$ and $v = w_1' + w_2'$, where $w_1, w_1' \in W$ and $w_2, w_2' \in W^{\perp}$. We want to show that $w_1 = w_1'$ and $w_2 = w_2'$. Now, $w_1 - w_1' = w_2' - w_2$. Because, we have $v = w_1 + w_2$ and $v = w_1' + w_2'$ then $w_1 + w_2 = w_1' + w_2'$. Therefore, $w_1 - w_1' = w_2' - w_2$. Also, $w_1 - w_1' \in W$, but $w_2' - w_2 \in W^{\perp}$. Thus, $w_1 - w_1' = w_2' - w_2 \in W \bigcap w^{\perp}$. But by theorem 6.11, $w_1 - w_1' = w_2' - w_2 = 0$. Hence, $w_1 = w_1'$ and $w_2 = w_2'$. Hence uniqueness part is proved.

Definition Let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, u_2, ..., u_k\}$, and let $v \in \mathbb{R}^n$. Then the **orthogonal projection of v onto** W is the vector

$$proj_W v = (v \cdot u_1)u_1 + ... + (v \cdot u_k)u_k.$$

If W is the trivial subspace of $\mathbb{R}^n$, then $proj_W v = 0$.

Note 1. The choice of orthonoraml basis for W in this definition is independent of choice of orthonormal basis for W. Thus if $\{z_1, z_2, ..., z_k\}$ is any other basis for W. then $proj_W v$ is equal to

$$proj_W v = (v \cdot z_1)z_1 + ... + (v \cdot z_k)z_k$$

.

This fact is illustrated in the following example

Example Consider the subspace $W = span(\{[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}], [\frac{4}{9}, \frac{4}{9}, \frac{7}{9}])$ of $\mathbb{R}^3$. Also consider $S = \{[4, 1, 1], [4, -5, -11]\}$. Then S is orthogonal. Let $v = [1, 2, 3]$. Then verify that the same vector for $proj_W v$ is obtained with the help of $B = \{u_1, u_2\} = \{[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}], [\frac{4}{9}, \frac{4}{9}, \frac{7}{9}]$ or with S.

Solution Clearly, B is orthonormal basis for W (How, Check it). Also, We have orthogonal set $S = \{[4, 1, 1], [4, -5, -11]\}$. Now Since $[4, 1, 1] = 3u_1 + 3u_2$ and $[4, -5, -1] = 9u_1 - 9u_2$, where u_1, u_2 are vector from B. Since, $|S| = dim(W) = 2$. Since, S is orthogonal. Therefore it is independent. Thus, S is orthogonal basis for W. Hence after normalizing the vectors -in S, we obtain the following second orthonoraml basis for W :

$$C = \{z_1, z_2\} = \left\{ \left[\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right], \left[\frac{4}{9\sqrt{2}}, -\frac{5}{9\sqrt{2}}, -\frac{11}{9\sqrt{2}}\right] \right\}$$

Now we will verify that the same vector for $proj_W v$ is obtained whether B or C is used as the orthonormal basis for W. Now using B yields

$$(v \cdot u_1)u_1 + (v \cdot u_2)u_2 = -\frac{2}{3}\left[\frac{8}{9}, -\frac{1}{9}, -\frac{4}{9}\right] + \frac{11}{3}\left[\frac{4}{9}, \frac{4}{9}, \frac{7}{9}\right] = \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{21}\right]$$

Similarly, using C gives

$$(v \cdot z_1)z_1 + (v \cdot z_2)z_2 = \frac{3}{\sqrt{2}}\left[\frac{4}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}\right] + \left(-\frac{13}{3\sqrt{2}}\right)\left[\frac{4}{9\sqrt{2}}, -\frac{5}{9\sqrt{2}}, -\frac{11}{9\sqrt{2}}\right] = \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{21}\right]$$

Hence, with either orthonormal basis we obtain $proj_W v = \left[\frac{28}{27}, \frac{46}{27}, \frac{85}{21}\right]$.

The definition of orthonoraml projection of a vector onto a subspace allows us to restate the projection Theorem as follows:

Theorem 2 *Let W be a subspace of $\mathbb{R}^n$. Then every vector v in $\mathbb{R}^n$ can be expressed in a unique way as $w_1 + w_2$, where $w_1 = proj_W v \in W$ and $w_2 = v - proj_W v \in W^\perp$. Moreover, W_2 can also be expressed as $proj_{W^\perp} v$.*

Exercise 1. Find the orthogonal projection of $v = [-1, 4, 3]$ onto the subspace W of $\mathbb{R}^3$ spanned by the orthogonal vectors $v_1 = [1, 1, 0]$ and $v_2 = [-1, 1, 0]$.

2. Consider the subspace $W = span(\{[1, -2, -1], [3, -1, 0]\}$ of $\mathbb{R}^3$. Let $v = [-1, 3, 2]$. Find $proj_W v$ and decompose v into $w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$. Is the decomposition is unique ?

Example Let W be the subspace of $\mathbb{R}^3$, whose vectors (beginning at the origin) lie in the plane $2x + y + z = 0$. Let $v = [-6, 10, 5]$. Find $proj_W v$ and decompose v into $w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$.

Solution We have $W = \{[x, y, z] \in \mathbb{R}^3 : 2x + y + z = 0\}$. We first notice that $[1, 0, -2]$ and $[0, 1, -1]$ are two linearly independent vectors in W. Hence, the are basis for W (Check it). Let $x_1 = [1, 0, -2]$ and $x_2 = [0, 1, -1]$.

Using the Gram Schmidt Process on these vector x_1 and x_2, We obtain the orthogonal basis $\{[1, 0, -2], [-2, 5, -1]\}$ for W (Verify it). After normalization, we have the orthonormal basis $\{u_1, u_2\}$ for W, where

$$u_1 = \left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right], \quad \textit{and} \quad u_2 = \left[-\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right]$$

Now,

$$\begin{aligned} w_1 = proj_W v &= (v \cdot u_1)u_1 + (v \cdot u_2)u_2 \\ &= -\frac{16}{\sqrt{5}}\left[\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right] + \frac{57}{\sqrt{30}}\left[-\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}\right] \\ &= \left[-\frac{16}{5}, 0, \frac{32}{5}\right] + \left[-\frac{114}{30}, \frac{285}{30}, -\frac{57}{30}\right] \\ w_1 &= \left[-7, \frac{19}{2}, \frac{9}{2}\right] \end{aligned}$$

Notice that $w_1 \in W$. Finally $w_2 = v - proj_W v$

$$[-6, 10, 5] - \left[-7, \frac{19}{2}, \frac{9}{2}\right] = \left[1, \frac{1}{2}, \frac{1}{2}\right]$$

i.e. $\quad w_2 = \left[1, \frac{1}{2}, \frac{1}{2}\right]$

Clearly, $w_2 \in W^\perp$, because it is orthogonal to both x_1 and x_2. Hence, we have decomposed $v = [-6, 10, 5]$ as

$$v = w_1 + w_2 = \left[-7, \frac{19}{2}, \frac{9}{2}\right] + \left[1, \frac{1}{2}, \frac{1}{2}\right]$$

Where $w_1 \in W$ and $w_2 \in W^\perp$.

Application: Distance from a Point to a subspace

Definition Minimum Distance

Let W be a subspace of $\mathbb{R}^n$, and assume all vectors in W have initial point at the origin. Let P be any point in $n-dimensional$ space. Then the Minimum Distance from P to W is the shortest distance between P and the terminal point of any vector in W.

The following theorem gives a formula for the minimum distance:

Theorem 3 *(Without Proof) Let W be a subspace of $\mathbb{R}^n$, and let P be a point in $n-dimensional$ space. If v is the vector from the origin to P, the the minimum distance from P to W is $||v - proj_W v||$.*

Note 1. Notice that the minimum distance can also be obtained using $||proj_{W^\perp} v||$.

Note 2. Notice if S is the terminal point of $proj_W v$, then $||v - proj_W v||$ represent the distance from P to S.

Example Let $W = \{[x, y, z] : 2x + y + z = 0\}$ be a subspace of $\mathbb{R}^3$. Find the minimum distance from the point $P(-6, 10, 5)$ to W.

Solution Let v be the vector from origin to the point $P(-6, 10, 5)$. Then the minimum distance from $P(-6, 10, 5)$ to W is $||v - proj_w v||$. In last Example we calculated that

$$v - proj_W v = \left[1, \frac{1}{2}, \frac{1}{2}\right]$$

Hence, the minimum distance from $P(-6, 10, 5)$ to W is

$$||v - proj_W v|| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}$$

Exercise Find the minimum distance from the point $P = (2, 3, -3, 1)$ to a subspace

$$W = span(\{[-1, 2, -1, 1], [2, -1, 1, -1]\})$$

in $\mathbb{R}^4$.