Study Material(I)

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1 Orthogonal Complements

In previous section we studied about orthogonal and orthonormal bases for $\mathbb{R}^n$ and also we learned uses of Gram-Schmidt process to replace a basis for a subspace of $\mathbb{R}^n$ with an orthogonal basis.

Now in this section we will extend the concept of orthogonal and orthonormal basis to orthogonal complement. Also, we study some elementary properties of orthogonal complements and investigate the orthogonal projection of a vector onto a subspace of $\mathbb{R}^n$.

1.1 Definition Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. The Orthogonal Complement of W, denoted by $W^\perp$ (Where $\perp$ is called perp, short for perpendicular complement), is the set of all vectors of $\mathbb{R}^n$ that are orthogonal to every vector in W. That is

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0, \forall w \in W\}$$

Theorem 1 *If W is a subspace of $\mathbb{R}^n$, then $v \in W^\perp$ if and only if v is orthogonal to every vector in a spanning set for W.*

Proof Let $S = \{w_1, w_2, ..., w_k\}$ be a spanning set for W. Let us suppose that $v \in W^\perp$. Then we must show that $v \cdot w = 0$ for $1 \leq i \leq k$. Since, $v \in W^\perp$ and by definition of $W^\perp$ we have $v \cdot w = 0$ for all $w \in W$. In particular $v \cdot w_i = 0$ for $1 \leq i \leq k$ and by the property of spanning set $S \subseteq span(s) = W$.

Conversely, assume that $v \cdot w_i = 0$ for $1 \leq i \leq k$. Let $w \in W = span(S)$. Then there exist real number $c_1, c_2, ..., c_k$ such that

$$w = c_1 w_1 + c_2 w_2 + ... + c_k w_k$$

$$Therefore, v \cdot = v \cdot (c_1 w_1 + c_2 w_2 + ... + c_k w_k)$$

$$= c_1(v \cdot w_1) + c_2(v \cdot w_2) + ... + c_k(v \cdot w_k) = 0, i.e. v \cdot w = 0 \implies v \in W^\perp$$

Example : 1 The orthogonal complement of $\mathbb{R}^n$ is $\{0\}$ $\downarrow$ singleton zero, since the zero vector is the only vector that is orthogonal to all of vectors in $\mathbb{R}^n$.

for the same reason, we have

$$\{0\}^{\perp} = \mathbb{R}^n.$$

Example : 2 Consider the subspace

$$W = \{[a,0,c] : a,c \in \mathbb{R}\} \text{ of } \mathbb{R}^3.$$

Note that W is spanned by the set $\{[1,0,0], [0,0,1]\}$, since

$$W = \{a[1,0,0] + b[0,0,1] : a,c \in \mathbb{R}\}$$
$$= \text{span}\{[1,0,0], [0,0,1]\}.$$

Hence, by theorem 1. a vector

$$[x,y,z] \in W^{\perp} \Leftrightarrow [x,y,z]\cdot[1,0,0] = 0$$
and
$$[x,y,z]\cdot[0,0,1] = 0$$

i.e. $[x,y,z] \in W^{\perp} \Leftrightarrow x = 0$ and $z = 0$.

$\Rightarrow W^{\perp} = \{[0,y,0] : y \in \mathbb{R}\} = \text{span}\{[0,1,0]\}$

$\Rightarrow \dim(W^{\perp}) = 1$

Notice that $W^{\perp}$ is a subspace of $\mathbb{R}^3$ of dimension 1 and that

$\dim(W) + \dim(W^{\perp}) = 2+1 = 3 = \dim(\mathbb{R}^3)$.

$\boxed{\text{Exercise:}}$ Let $W = \{a[-3, 2, 4] \mid a \in \mathbb{R}\}$. Why W is a subspace of $\mathbb{R}^3$? Also, find $W^{\perp}$ and verify that

$$\dim(W) + \dim(W^{\perp}) = \dim(\mathbb{R}^3).$$

* Properties of orthogonal complements

Theorem.2 Let W be a subspace of $\mathbb{R}^n$. Then $W^{\perp}$ is a subspace of $\mathbb{R}^n$, and $W \cap W^{\perp} = \{0\}$.

Proof: We know the zero vector is orthogonal to every vector in $\mathbb{R}^n$!
Hence, $W^{\perp} \neq \phi$

Now, we need to show that if $x, x_1, x_2 \in W^\perp$ and c be a scalar. Then

(i) $x_1 + x_2 \in W^\perp$

(ii) $cx \in W^\perp$.

(i) consider $(x_1+x_2)\cdot w$ for all $w \in W$.

$$(x_1+x_2)\cdot w = x_1\cdot w + x_2\cdot w = 0+0=0$$

$(\because x_1, x_2 \in W^\perp)$

$\Rightarrow x_1+x_2 \in W^\perp$

(ii) similarly, let x be a vector in $W^\perp$ and let c be any scalar. We must show that $cx \in W^\perp$. However for all $w \in W$,

$$c(x)\cdot w = c\cdot(x\cdot w) = c(0) = 0$$

$\Rightarrow cx \in W^\perp$. $(\because x \in W^\perp)$

Hence, $W^\perp$ is a subspace of $\mathbb{R}^n$.

finally, we show that $W \cap W^\perp = \{0\}$

let $w \in W \cap W^\perp$.

Then $w \in W$ and $w \in W^\perp$. and hence,

$w\cdot w = 0$ This, only possible when w itself is a zero vector.

$\Rightarrow w = 0$

Hence, proved.

Theorem 2. (Without proof):

Let W be a subspace of $\mathbb{R}^n$. Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal basis for W contained in an orthogonal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $\mathbb{R}^n$. Then $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is an orthogonal basis for $W^\perp$.

Corollary. 3 Let W be a subspace of $\mathbb{R}^n$. Then

$$\dim(W) + \dim(W^\perp) = n = \dim(\mathbb{R}^n).$$

Proof: Let W be a subspace of $\mathbb{R}^n$ of dimension k. By ~~Theorem~~ Gram-Schmidt process, W has an orthogonal basis $\{v_1, v_2, \ldots, v_k\}$.

We thus have an orthogonal set $\{v_1, v_2, \ldots, v_k\}$ of non-zero vectors in $\mathbb{R}^n$. Hence, by theorem 6.5 of section 6.1, it can be enlarged to an orthogonal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for $\mathbb{R}^n$.

Then by theorem 2, $\{v_{k+1}, \ldots, v_n\}$ is an orthogonal basis for $W^\perp$ and so, $\dim(W^\perp) = n-k$.

Hence, $\dim(W) + \dim(W^\perp) = n = \dim(\mathbb{R}^n)$.

Example: Let $W = \text{span}(\{[1,2,3]\})$.

Then W is a subspace of $\mathbb{R}^3$ and $\dim(W) = 1$.

Now $W^\perp = \{[x,y,z] \in \mathbb{R}^3 : [x,y,z]\cdot[1,2,3] = 0\}$

$= \{[x,y,z] \in \mathbb{R}^3 : x+2y+3z=0\}$

Thus, $W^\perp$ is the set of all vector $[x,y,z]$ lying in the plane $x+2y+3z=0$.

and $\dim(W^\perp) = 2$.

$\Rightarrow \dim(W) + \dim(W^\perp) = 1+2 = 3.$

Corollary: 4 Let W be a subspace of $\mathbb{R}^n$.
Then $(W^\perp)^\perp = W$.

Proof: Trivially, $W \subseteq (W^\perp)^\perp$ — (1)

Thus tho show that $W = (W^\perp)^\perp$.
it is enough to show that
$\dim(W) = \dim(W^\perp)^\perp$.

By ~~theorem~~ corollary 3.,
we have

$(\dim(W^\perp)^\perp) = n - \dim(W^\perp) = \dim(W)$.

i.e. $\dim((W^\perp)^\perp) = \dim(W)$

Hence, By property of dimension.
$(W^\perp)^\perp = W$.

Example: for the subspace

$$W = \{[x, y, z] \in \mathbb{R}^3 : 3x - y + 4z = 0\}$$

of $\mathbb{R}^3$. Find the orthogonal complement $W^\perp$ and verify that $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^3)$.

Soln: The subspace W is collection of all vectors $[x, y, z]$ lying in the plane $3x - y + 4z = 0$ i.e. $[x, y, z] \cdot [3, -1, 4] = 0$

Thus, W is the set of all vectors orthogonal to $[3, -1, 4]$, and hence orthogonal to the subspace

$$Y = \text{span}(\{[3, -1, 4]\}) \text{ of } \mathbb{R}^3.$$

So, by defn. of orthogonal complement,

$W = Y^\perp$.

Hence, $W^\perp = (Y^\perp)^\perp = Y = \text{span}(\{[3, -1, 4]\})$

Notice that $\dim(W) = 2$.

$\because$ W is spanned by two L.I. vectors $[1, 3, 0]$ and $[-4, 0, 3]$ in W.

$\therefore$ $\dim(W) + \dim(W^\perp) = 1 + 2 = 3 = \dim(\mathbb{R}^3)$.