Lecture 3

(Binomial Distribution)

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0.1 Important points regarding Binomial Distribution

(a). Number of trials (n) is finite (say *5, 6, 7, ...*).

(b). In each trial, probability of event *X* is p and probability of failure of *X* is $q(=1-p)$.
[p is also called probability of success]

(c). Each trial is independent and is done in identical conditions and circumstances.

(d). Probability of *r* success is given by
$P(r) = \binom{n}{r} p^r q^{n-r}$

0.2 Expectation (or Mean) of Binomial Ditribution

Mean of Binomial distribution is defined as

$$\bar{x} = \frac{\sum_{r=0}^{n} r\,P(r)}{\sum_{r=0}^{n} P(r)} = \sum_{r=0}^{n} r\,P(r) \qquad \left[\because \sum_{r=0}^{n} P(r) = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} = (p+q)^n = 1^n = 1\right]$$

$$= \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \binom{n}{r} \left(p\frac{\partial p^r}{\partial p}\right) q^{n-r} \qquad \left[\because \frac{\partial}{\partial p} p^r = r\,p^{r-1}\right]$$

$$= p\frac{\partial}{\partial p} \sum_{r=0}^{n} \binom{n}{r} p^r\, q^{n-r}$$

$$= p\frac{\partial}{\partial p}(p+q)^n$$

$$= p\,n\,(p+q)^{n-1}$$

$$= n\,p\,(1)^{n-1} \qquad [\because p+q=1]$$

$$= n\,p$$

$\bar{x}$ is also called expectation of random variable X i.e., $\bar{x}$ = $E(X)$.

Note: $E(X)$ is also known as first moment, denoted by μ_1.

0.3 Second Moment of Binomial Distribution

Second Moment of Binomial distribution is given by

$$
\begin{aligned}
E(X^2) &= \frac{\sum_{r=0}^{n} r^2\, P(r)}{\sum_{r=0}^{n} P(r)} = \sum_{r=0}^{n} r^2\, P(r) \qquad \left[\because \sum_{r=0}^{n} P(r) = \sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} = (p+q)^n = 1^n = 1\right] \\
&= \sum_{r=0}^{n} r^2 \binom{n}{r} p^r q^{n-r} \\
&= \sum_{r=0}^{n} \binom{n}{r} \left(p^2 \frac{\partial^2}{\partial p^2} p^r + r\, p^r\right) q^{n-r} \quad \left[\because \frac{\partial^2}{\partial p^2} p^r = r(r-1)\, p^{r-2} \implies p^2 \frac{\partial^2}{\partial p^2} p^r = r(r-1)\, p^r\right] \\
&= p^2 \frac{\partial^2}{\partial p^2} \sum_{r=0}^{n} \binom{n}{r} p^r\, q^{n-r} + \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r} \\
&= p^2 \frac{\partial^2}{\partial p^2} (p+q)^n + n\, p \\
&= p^2\, n\, (n-1)\, (p+q)^{n-2} + n\, p \\
&= n\, (n-1)\, p^2\, (1)^{n-2} + n\, p \qquad [\because p+q=1] \\
&= n\, p\, [1 + (n-1)p]
\end{aligned}
$$

$E(X^2)$ is also known as second moment, denoted by μ_2.

0.4 Variance of Binomial Distribution

Variance of Binomial distribution is given by

$$
\begin{aligned}
\sigma^2 &= \mu_2 - \mu_1^2 \\
&= n\, p\, [1 + (n-1)p] - n^2\, p^2 \\
&= n\, p - n\, p^2 \\
&= n\, p\, (1-p) \\
&= n\, p\, q
\end{aligned}
$$

0.5 Standard Deviation of Binomial Distribution

Standard Deviation of Binomial Distribution is given by

$$\sigma = \sqrt{\sigma^2} = \sqrt{n\, p\, q}$$

0.6 Summary (Binomial Distribution)

Mean or Expectation	$E = n\,p$
Variance	$\sigma^2 = n\,p\,q$
Standard Deviation	$\sigma = \sqrt{n\,p\,q}$

0.7 Examples

Example 1. Find the probability of (a) 2 or more heads, (b) fewer than 4 heads, in a single toss of 6 fair coins.

Solution Let X be the number of heads in single toss of 6 fair coins. Then we have, $n = 6$, $p = 1/2$ (Probability of getting head) and $q = 1/2$.

(a) We have to compute $P(X \geq 2)$

$$\begin{aligned} P(X \geq 2) &= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) \\ &= \binom{6}{2}\left(\frac{1}{2}\right)^6 + \binom{6}{3}\left(\frac{1}{2}\right)^6 + \binom{6}{4}\left(\frac{1}{2}\right)^6 + \binom{6}{5}\left(\frac{1}{2}\right)^6 + \binom{6}{6}\left(\frac{1}{2}\right)^6 \\ &= \frac{57}{64} \end{aligned}$$

(b) We have to compute $P(X < 4)$

$$\begin{aligned} P(X < 4) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= \binom{6}{0}\left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^6 + \binom{6}{2}\left(\frac{1}{2}\right)^6 + \binom{6}{3}\left(\frac{1}{2}\right)^6 \\ &= \frac{21}{32} \end{aligned}$$

Example 2. If X denotes the number of heads in a single toss of 4 fair coins, find (a) $P(X = 3)$, (b) $P(X < 2)$, (c) $P(X \leq 2)$, (d) $P(1 < X \leq 3)$.

Solution Here $n = 4$, $p = 1/2$ and $q = 1/2$.

(a)We have to compute $P(X = 3)$,

$$\begin{aligned} P(X = 3) &= \binom{4}{3}\left(\frac{1}{2}\right)^4 \\ &= \frac{1}{4} \end{aligned}$$

(b) We have to compute $P(X < 2)$,

$$\begin{aligned} P(X < 2) &= P(X = 0) + P(X = 1) \\ &= \binom{4}{0}\left(\frac{1}{2}\right)^4 + \binom{4}{1}\left(\frac{1}{2}\right)^4 \\ &= \frac{5}{16} \end{aligned}$$

(c) We have to compute $P(X \leq 2)$,

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \binom{4}{0}\left(\frac{1}{2}\right)^4 + \binom{4}{1}\left(\frac{1}{2}\right)^4 + \binom{4}{2}\left(\frac{1}{2}\right)^4 \\ &= \frac{11}{16} \end{aligned}$$

(d) We have to compute $P(1 < X \leq 3)$,

$$\begin{aligned} P(1 < X \leq 3) &= P(X = 2) + P(X = 3) \\ &= \binom{4}{2}\left(\frac{1}{2}\right)^4 + \binom{4}{3}\left(\frac{1}{2}\right)^4 \\ &= \frac{5}{8} \end{aligned}$$

Example 3. Out of 800 families with 5 children each, how many would you expect to have (a) 3 boys, (b) 5 girls, (c) either 2 or 3 boys? Assume equal probabilities for boys and girls.

Solution Let X denotes the number of boys, p be the probability that a child is boy and q that a child is girl. Here, $n = 5$, $p = 1/2$ and $q = 1/2$.

(a) We have to compute $P(X = 3)$

$$\begin{aligned} P(X = 3) &= \binom{5}{3}\left(\frac{1}{2}\right)^5 \\ &= \frac{5}{16} \end{aligned}$$

Therefore, the number of families with 3 boys are $\frac{5}{16} \times 800 = 250$.

(b) We have to compute $P(X = 0)$

$$\begin{aligned} P(X = 0) &= \binom{5}{0}\left(\frac{1}{2}\right)^5 \\ &= \frac{1}{32} \end{aligned}$$

Therefore, the number of families with 5 girls are $\frac{1}{32} \times 800 = 125$.

(c) We have to compute $P(X = 2) + P(X = 3)$

$$\begin{aligned} P(X = 2) + P(X = 3) &= \binom{5}{2}\left(\frac{1}{2}\right)^5 + \binom{5}{3}\left(\frac{1}{2}\right)^5 \\ &= \frac{5}{8} \end{aligned}$$

Therefore, the number of families with either 2 or 3 boys are $\frac{5}{8} \times 800 = 500$.

Example 4. Compute the (a) mean, (b) variance, (c) standard deviation for a binomial distribution in which $p = 0.7$ and $n = 60$.

Solution (a) The mean is given by,

$$\begin{aligned} \mu &= n\,p \\ &= 60 \times 0.7 \\ &= 42 \end{aligned}$$

(b) The variance is given by,

$$\begin{aligned} \sigma^2 &= n\,p\,q \\ &= 60 \times 0.7 \times 0.3 \qquad\qquad [\because q = 1 - p = 0.3] \\ &= 12.6 \end{aligned}$$

(c) The standard deviation is given by,

$$\begin{aligned} \sigma &= \sqrt{n\,p\,q} \\ &= \sqrt{60 \times 0.7 \times 0.3} \qquad\qquad [\because q = 1 - p = 0.3] \\ &= \sqrt{12.6} \\ &\approxeq 3.55 \end{aligned}$$

0.8 Problems

1. Consider the Binomial distribution $B(n, p)$. Show that
   (a) $\frac{P(k)}{P(k-1)} = \frac{(n-k+1)p}{kq}$.
   (b) $P(k-1) < P(k)$ for $k < (n+1)p$ and $P(k-1) > P(k)$ for $k > (n+1)p$.

2. Find the probability of getting a total of 11 (a) once, (b) twice, in a two tosses of pair of fair dice.

3. Out of 2000 families with 4 children each, how many would you expect to have (a) at least 1 boy, (b) 2 boys, (c) 1 or 2 girls, (d) no girls?

4. A card is drawn and replaced in an ordinary 52-card deck. Find the number of times a card must be drawn so that
   (a) there is an even chance of drawing a heart.
   (b) the probability of drawing a heart exceeds 0.75.

5. A box contains 3 red marbles and 2 white marbles. A marble is drawn and replaced 3 times from the box. Find the probability that:

(a) 1 red marble was drawn,

(b) 2 red marbles were drawn,

(c) at least 1 red marble was drawn.

6. Find the probability of guessing correctly at least 6 of the 10 answers on a true-false examination.

7. An insurance sales representative sells policies to 5 men, all of identical age and in good health. According to the actuarial tables, the probability that a man of this particular age will be alive 30 years hence is . Find the probability that in 30 years (a) all 5 men, (b) at least 3 men, (c) only 2 men, (d) at least 1 man will be alive.

8. A certain type of missile hits its target with probability $p = 1/5$. (a) If 3 missiles are fired, find the probability that the target is hit at least once. (b) Find the number of missiles that should be fired so that there is at least a 90 percent probability of hitting the target (at least once).

9. Suppose 2 percent of the bolts produced by a factory are defective. In a shipment of 3600 bolts from the factory, find the expected number *E* of defective bolts and the standard deviation σ

10. Team *A* has probability 0.4 of winning whenever it plays (and there are no ties). Let Xdenote the number of times *A* wins in four games.

    (a) Find the distribution of X.

    (b) Find the mean, variance and standard deviation of X.

11. Let X be a binomially distributed random variable with $E(X) = 2$ and $Var(X) = 4/3$. Find n and p.

12. Evaluate (a) $\sum(x - \mu)^3 P(x)$, (b) $\sum(x - \mu)^3 P(x)$ for the binomial distribution.

0.9 References

(a). Introduction to Probability and Statistics, 2016 by Seymour Lipschutz and John J. Schiller.

(b). Probability and Statistics, 3e by Murray R. Spiegel, John J. Schiller and R. Alu Srinivasan.