<u>Remark</u> :- For $A \in M_{n\times n}(\mathbb{F})$. The characteristic polynomial of A is a polynomial of degree n.

<u>Theorem 5.3</u> Let $A \in M_{n\times n}(\mathbb{F})$. Then

(a) The characteristic polynomial of A is a polynomial of deg 'n' with leading coefficient $(-1)^n$.

(b) A has at most 'n' distinct eigenvalues.

<u>Proof</u> Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \in M_{n\times n}(\mathbb{F})$.

(a) Characteristic polynomial of A is $\det(A - \lambda I_n)$

$$= \det\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \; \lambda \end{bmatrix} \right)$$

$$= \det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

It is clear that (a) holds.

(b) Since deg of characteristic poly. is 'n' $\therefore$ By theorem 5.2, ~~characteristic~~ $\det(A - \lambda I_n) = 0$ has atmost 'n' ~~roots~~ distinct roots $\Longleftrightarrow$ A has atmost 'n' distinct eigenvalues.

Theorem 5.4 Let $T : V \to V$ be a linear operator and λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to 'λ' $\iff$ $v \neq 0$ and $v \in N(T - \lambda I)$.

Proof Let 'v' $\in V$ be an eigen vector of T corresponding to 'λ' $\iff$ $v \neq 0$ and

$$T(v) = \lambda I$$

$$\iff (T - \lambda I)\, v = 0 \quad \text{and} \quad v \neq 0$$

$$\iff v \in N(T - \lambda I) \quad \text{and} \quad v \neq 0$$

Ex-6 Find eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

~~Do it yourself~~.

Sol^n The characteristic equation of A is

$$\det(A - \lambda I_2) = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow \lambda^2 + 1 - 2\lambda - 4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + \lambda - 3 = 0$$

$$\Rightarrow \lambda(\lambda - 3) + 1(\lambda - 3) = 0 \quad \Rightarrow \lambda = 3, -1$$

Let $\lambda_1 = 3$, $\lambda_2 = -1$

Let $v_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the eigen vector of A corresponding to eigen value $\lambda_1 = 3$

$\Rightarrow \quad Av_1 = \lambda_1 v_1 \qquad \Rightarrow \quad Av_1 = 3v_1$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x+y \\ 4x+y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

$\Rightarrow \quad x + y = 3x \quad \& \quad 4x + y = 3y$

$\Rightarrow \quad \boxed{y = 2x}$

$\Rightarrow \quad v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Eigen space $E_3 = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x \in \mathbb{R} \right\}$

Similarly $E_{-1} = \left\{ x \begin{bmatrix} 1 \\ -2 \end{bmatrix} : x \in \mathbb{R} \right\}$

<u>To show</u>:- $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ forms a basis for $\mathbb{R}^2$.

Since $\dim \mathbb{R}^2 = 2$ $\therefore$ it is enough to show that $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ is Linearly Independent. [By COR of Replacement thm]

let $a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$ for some $a, b \in \mathbb{R}$

$\Rightarrow \begin{bmatrix} a+b \\ 2a-2b \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} a+b = 0 \\ 2a-2b = 0 \end{matrix} \quad \Rightarrow \quad a = 0 = b$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$ forms a basis ...

Theorem (A)

Let T be a linear operator on an n-dimensional vector space and let $[T]_\beta = A$ where β is an ordered basis for V. For $v \in V$, let $\phi_\beta : V \to \mathbb{F}^n$ be defined as

$$\phi_\beta(v) = [v]_\beta \text{, the co-ordinate vector}$$

of v relative to β. Then, for an eigenvalue λ of A (and hence of T), a ~~good~~ vector $y \in \mathbb{F}^n$ is an eigenvector of A corresponding to $\lambda \Leftrightarrow \phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

Proof: Let v be an eigenvector of T corresponding to λ. $\Rightarrow Tv = \lambda v$ and $v \neq 0$

~~Since ϕ_β is an isomorphism $\Rightarrow \exists\ y \in \mathbb{F}^n$ s.t.~~

To show: $\phi_\beta(v)$ is an eigenvector of A corresponding to λ.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & V \arrow[d, "\phi_\beta"] \\
\mathbb{F}^n \arrow[r, "A"', "{=[T]_\beta}" below] & \mathbb{F}^n
\end{tikzcd}$$

$\Downarrow$

$$A\phi_\beta = \phi_\beta T$$

Consider

$$A\phi_\beta(v) = L_A\,\phi_\beta(v)$$
$$= \phi_\beta T(v) = \phi_\beta(\lambda v)$$
$$= \lambda\,\phi_\beta(v)$$
$$\Rightarrow A\phi_\beta(v) = \lambda\,\phi_\beta(v)$$

Since $v \neq 0 \Rightarrow \phi_\beta(v) \neq 0$

($\because \phi_\beta$ is automorphism and so is 1-1]

$\therefore \phi_\beta(v)$ is an eigenvector of A corresponding to λ

By reversing the steps, the other implication also holds.

Ex-7 Let T be a linear operator on $P^2(\mathbb{R})$ defined as
$$T(f(x)) = f(x) + (x+1)f'(x) \quad (\text{Defined in Ex-5})$$

We have found in Ex-5 that T has eigen values $1, 2$ and 3 and $A = [T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

Let $\lambda_1 = 1$. Let $y_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an Eigen vector of A

wrt eigen value $\lambda_1 = 1$.

$\Rightarrow (A - \lambda_1 I) y_1 = 0 \Rightarrow Ay_1 = \lambda_1 y_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+y \\ 2y+2z \\ 3z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{aligned} x+y &= x \\ 2y+2z &= y \\ 3z &= z \end{aligned}$$

$\Rightarrow y = 0, \quad z = 0$. Let $x = a$

$$\Rightarrow y_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad a \neq 0$$

$$\Rightarrow \phi_\beta^{-1}(y_1) = a \cdot 1 + 0 \cdot X + 0 \cdot X^2$$
$$= a \cdot 1 = a$$

$\therefore$ By theorem (A), 'a' $(\neq 0)$ is an eigen vector of T wrt eigen value $\lambda_1 = 1$.

Let $\lambda_2 = 2$ and $y_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the corresponding eigen vector of A.

$\Rightarrow (A - \lambda_2 I) y_2 = 0$

$$\Rightarrow \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \quad -x + y = 0$$
$$2z = 0$$
$$z = 0$$

$$\Rightarrow \quad x = y \quad \& \quad z = 0$$

$$\Rightarrow \quad y_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

let $x = a \quad \Rightarrow \quad y_2 = \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}$

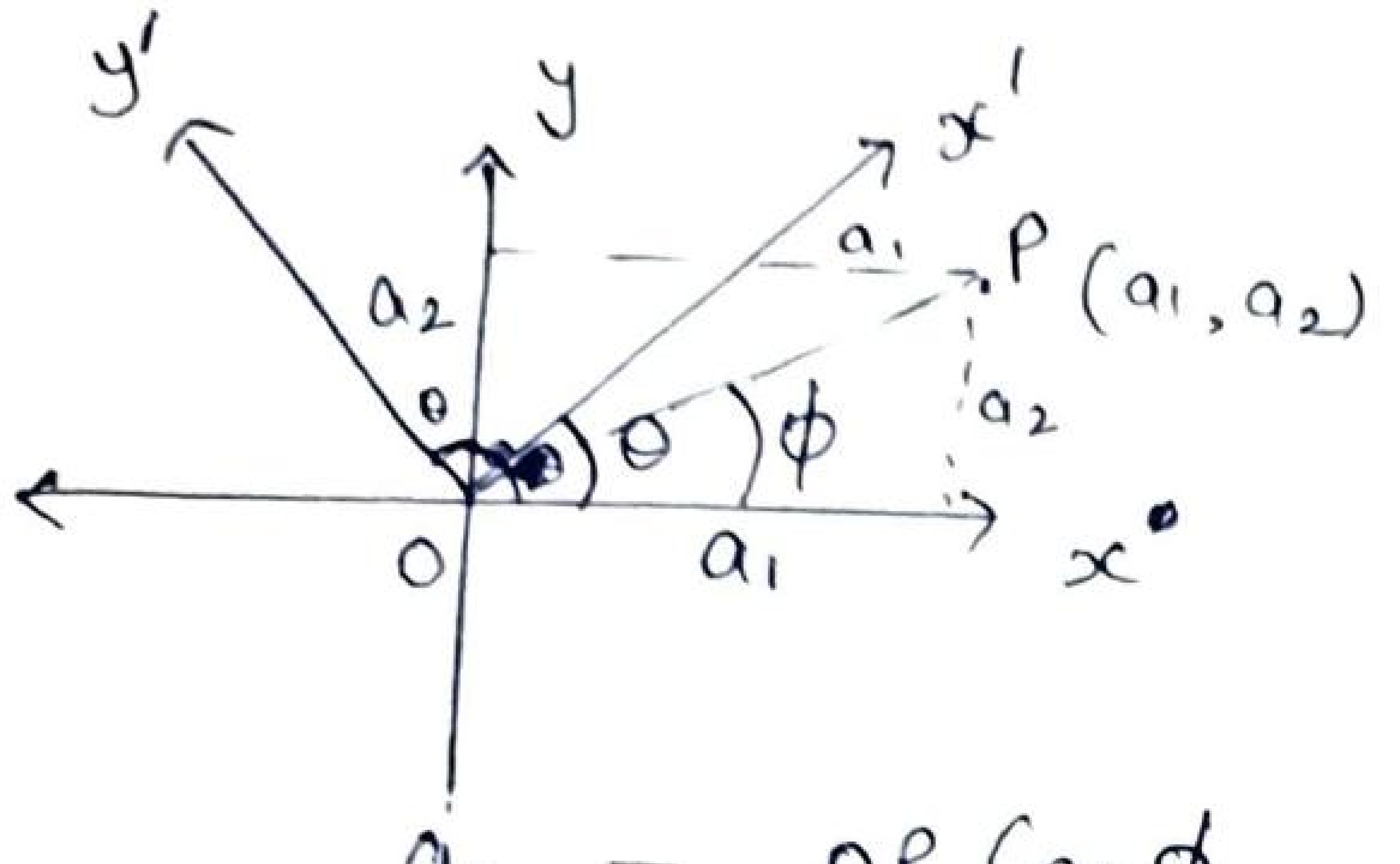
$$\Rightarrow \quad \phi_\beta^{-1}(y_2) = a.1 + a.X + 0.X^2$$
$$= a + aX = a(1+X), \quad a \neq 0$$

be the eigen vector of T corresponding to $\lambda_2 = 2$.

Ex- let T be an operator on $\mathbb{R}^2$ that rotates the plane $\mathbb{R}^2$ through the angle θ where $0 < \theta < \pi$.

~~Sol~~ let $P(a_1, a_2)$ be a point in $\mathbb{R}^2$ and $P'(a_1', a_2')$ be its image under T.

$$\Rightarrow \quad \frac{a_1}{OP} = \cos\phi \quad \& \quad \frac{a_2}{OP} = \sin\phi$$



$$a_1 = OP\cos\phi$$

$$a_2 = OP\sin\phi \qquad \cancel{\tan\phi = \frac{a_2}{a_1}}$$

$$a_1' = OP\cos(\theta+\phi)$$

$$a_2' = OP\sin(\theta+\phi)$$

(Since T rotates pt P by θ angle)

$$a_1' = OP(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= OP\left(\cos\theta\cdot\frac{a_1}{OP} - \sin\theta\cdot\frac{a_2}{O\cdot P}\right)$$

$$= a_1\cos\theta - a_2\sin\theta$$

$$a_2' = OP(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

$$= OP\left(\sin\theta\cdot\frac{a_1}{OP} + \cos\theta\cdot\frac{a_2}{O\cdot P}\right)$$

$$= a_1\sin\theta + a_2\cos\theta.$$

$$\Rightarrow T(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta,\ a_1\sin\theta + a_2\cos\theta)$$

$$\forall (a_1, a_2) \in \mathbb{R}^2.$$

~~Consider~~ ~~Characteristic polynomial~~ of T_θ is

~~det $(T_\theta - \lambda I)$ = det~~

Let $e_1 = (1,0)$ and $e_2 = (0,1)$, $\beta = \{e_1, e_2\}$ forms a basis for $\mathbb{R}^2$.

$$T(e_1) = (\cos\theta, \sin\theta)$$
$$= \cos\theta \cdot e_1 + \sin\theta \cdot e_2$$
$$T(e_2) = (-\sin\theta, \cos\theta)$$
$$= -\sin\theta \cdot e_1 + \cos\theta \cdot e_2$$

$$\Rightarrow [T]_\beta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation of $[T]_\beta$ is

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\Rightarrow \cos^2\theta + \sin^2\theta + \lambda^2 - 2\lambda\cos\theta = 0$$

$$\Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm 2\sqrt{-\sin^2\theta}}{2}$$

$$\Rightarrow \lambda = \frac{\cos\theta \pm i\sin\theta}{1} \equiv \text{Complex numbers}$$

$\Rightarrow T$ has no eigen values.

Observe that for any non-zero vector v and $T(v)$ are not collinear and $\therefore$ it is directly implies that T has no eigen values.

Ex - Try yourself and ask if problems occur.